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Lower signalizer lattices in alternating and symmetric groups

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Abstract. We prove that the subgroup lattices of finite alternating and symmetric groups do not contain so-called lower signalizer lattices in the class $D\Delta$. This result is one step in a program to show that the lattices in the class $D\Delta$ are not isomorphic to an interval in the subgroup lattice of any finite group.

In 1980 in [13], Palfy and Pudlak proved that the following two statements are equivalent:

(1) Each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of a finite group.

(2) Each finite lattice is isomorphic to the lattice of congruences of some finite algebra.

Given a group G and a subgroup H of G, write $\mathcal{O}_G(H)$ for the lattice of overgroups of H in G, and define a *finite subgroup interval lattice* to be a lattice of the form $\mathcal{O}_G(H)$ for some finite group G and subgroup H of G. In the thirty years since [13] appeared, the question of whether each finite lattice is a finite subgroup interval lattice has remained open and of significant interest. The general consensus seems to be that the answer is negative, and indeed that finite subgroup interval lattices constitute a small subclass of the class of all finite lattices.

Early work on the question centered on so-called M-lattices, the lattices of depth 2. Probably the high point of this effort is the reduction (in 1997 by Baddeley and Lucchini in [10]) of the existence of M-lattices as finite group interval lattices, to various questions about finite simple groups.

Each finite lattice Λ has a greatest member ∞ and least member 0. Set $\Lambda' = \Lambda - \{0, \infty\}$. Given a positive integer m, write $\Delta(m)$ for the lattice of subsets of an m-set. Given finite lattices Λ_1 and Λ_2, write $\Lambda_1 * \Lambda_2$ for the lattice Λ such that Λ'

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is the disjoint union of Λ_1' and Λ_2', with no member of Λ_1' comparable to a member of Λ_2'. Given integers $t > 1$ and $m_1, \dots, m_t > 2$, define $D\Delta(m_1, \dots, m_t)$ to be the lattice $\Delta(m_1) * \cdots * \Delta(m_t)$.

In 2003 in [15], Shareshian made a series of conjectures related to the Palfy–Pudlack Question. Write $\Delta(H, G)$ for the order complex of the poset $\mathcal{O}_G(H)'$. Shareshian's strongest conjecture asserts that for each finite group G and subgroup H, the complex $\Delta(H, G)$ has the homotopy type of a wedge of spheres. His weakest conjecture asserts that $D\Delta(3, 3)$ is not a finite subgroup interval lattice.

Inspired by these conjectures, Shareshian advanced the following conjecture, which sits between his other two conjectures:

Conjecture $D\Delta$. Given integers $t > 1$ and $m_1, \dots, m_t > 2$, there does not exist a finite group G and subgroup H of G such that the overgroup lattice $\mathcal{O}_G(H)$ is isomorphic to $D\Delta(m_1, \dots, m_t)$.

In Theorem 2 of [6], the author proves:

Reduction Theorem for $D\Delta$. *Assume Λ is $D\Delta(m_1, \dots, m_t)$-lattice for some integers $t > 1$ and $m_i > 2$. Assume further that Λ is a finite subgroup interval lattice. Then there exists an almost simple group G such that either*

(1) $\Lambda \cong \mathcal{O}_G(H)$ *for some* $H \leq G$, *or*

(2) *there exists a nonabelian finite simple group L and $\gamma = (G, H, J) \in \mathcal{T}(L)$, such that $G = \langle \mathcal{W}_0(\gamma), H \rangle$ and $\Lambda \cong \Xi(\gamma)$.*

Recall G is *almost simple* if G has a unique minimal normal subgroup $F^*(G)$, and $F^*(G)$ is a nonabelian simple group. Further $\mathcal{T}(L)$ is the collection of triples $\gamma = (G, H, J)$ such that G is a finite group, $H \leq G$, and $J \trianglelefteq H$ with

$$F^*(H/J) \cong L.$$

Write $\mathcal{W} = \mathcal{W}(\gamma)$ for the set of *signalizers* for H in G; that is those H-invariant subgroups W of G such that $H \cap W = J$. Given $G_0 \trianglelefteq G$, set $\hat{G} = G_0 J$ and $\mathcal{W}_0 = \mathcal{W}_0(\gamma, G_0) = \{W \in \mathcal{W} : W \leq \hat{G}\}$, and partially order $\mathcal{W}_0$ by inclusion. The poset $\Xi(\gamma, G_0)$ is obtained by adjoining a greatest member ∞ to the poset $\mathcal{W}_0$. It turns out (cf. 2.4) that $\Xi(\gamma, G_0)$ is a lattice, called a *lower signalizer lattice*. In the Reduction Theorem, $\mathcal{W}_0(\gamma) = \mathcal{W}_0(\gamma, F^*(G))$ and $\Xi(\gamma) = \Xi(\gamma, F^*(G))$.

The Reduction Theorem reduces Conjecture $D\Delta$ to two questions about sublattices of the lattice of subgroups of almost simple groups. The obvious first test cases for the two questions are the alternating and symmetric groups.

In [8], John Shareshian and the author prove that if G is alternating or symmetric, and $H \leq G$, then $\mathcal{O}_G(H)$ is not a $D\Delta$-lattice. This paper treats the lower signalizer lattice case by proving:

Theorem A. *Let G be a finite alternating or symmetric group and $H \leq G$. Let $t > 1$ and $m_1, \ldots, m_t > 2$ be integers. Then there does not exist a finite simple group L and $J \trianglelefteq H \leq G$ such that $\gamma = (G, H, J) \in \mathcal{T}(L)$, $G = \langle \mathcal{W}_0(\gamma), H \rangle$, and $\Xi(\gamma) \cong D\Delta(m_1, \ldots, m_t)$.*

The proof of Theorem A is divided into three cases: H primitive; H transitive but imprimitive; H intransitive. The three cases are treated in Theorem 5.23 in Section 5, Theorem 9.5 in Section 9, and Theorem 10.39 in Section 10.

Results in Sections 2 and 3 on lower signalizer lattices are also of independent interest.

See [1] for terminology and notation involving finite groups.

1 Notation, terminology, and preliminary lemmas

In this section we list some more specialized notation, some of which is used only in this paper. We also prove a number of preliminary lemmas.

Let G be a group and X and J subgroups of G. Define

$$\mathcal{I}_G(J) = \{H \leq G : J \leq N_G(H)\},$$

and for $X \in \mathcal{I}_G(J)$, set $\mathcal{V}_X(J) = \mathcal{I}_X(J) \cap \mathcal{O}_X(J \cap X)$. Define $\ker_J(G)$ to be the largest normal subgroup of G contained in J. Write $\mathcal{M}(J) = \mathcal{M}_G(J)$ for the set of maximal overgroups of J in G; that is $\mathcal{M}(J)$ is the set of maximal members of the poset $\mathcal{O}_G(J) - \{G\}$.

Suppose that Ω is a finite set, and set $S = \mathrm{Sym}(\Omega)$ the symmetric group on Ω and $A = \mathrm{Alt}(\Omega)$ the alternating group on Ω. Let $G \leq S$.

Write $\mathrm{Fix}(G) = \mathrm{Fix}_\Omega(G)$ for the set of fixed points of G on Ω, and $\mathrm{Mov}(G) = \mathrm{Mov}_\Omega(G)$ for the set $\Omega - \mathrm{Fix}(G)$ of points of Ω moved by G.

For $\Gamma \subseteq \Omega$, write G_Γ for the pointwise stabilizer in G of Γ; that is

$$G_\Gamma = \{g \in G : \Gamma \subseteq \mathrm{Fix}(G)\}.$$

Write $N_G(\Gamma)$ for the global stabilizer in G of Γ, and write G^Γ for the image in $\mathrm{Sym}(\Gamma)$ of $N_G(\Gamma)$ under the restriction map $g \mapsto g_{|\Gamma}$.

Let $\mathcal{P} = \mathcal{P}(\Omega)$ be the set of partitions of Ω. Partially order $\mathcal{P}$ by $Q \leq P$ if P is a refinement of Q. Thus $\mathcal{P}$ is a lattice with the obvious operators $\wedge$ and $\vee$ described in Section 3 of [5]. In particular $\mathcal{P}$ has a least member 0 and greatest member ∞, and, as in the introduction, $\mathcal{P}' = \mathcal{P} - \{0, \infty\}$. Let $\mathcal{P}(G)$ be the set of G-invariant partitions of Ω, and $\mathcal{P}'(G) = \mathcal{P}' \cap \mathcal{P}(G)$.

For $Q \leq P$ in $\mathcal{P}$ and $B \in Q$, write P_B for the partition $\{C \in P : C \subseteq B\}$ of B.

Given a partition $\Gamma \in \mathcal{P}$, let $N_G(\Gamma)$ be the subgroup of G acting on Γ. Then G_Γ is the kernel of the action of $N_G(\Gamma)$ on Γ. Write $K_+(\Gamma)$ for the subgroup of A generated by the groups $A_{\Omega-\gamma}$, $\gamma \in \Gamma$. Thus $K_+(\Gamma)$ is isomorphic to a direct product of alternating groups.

1.1. *Let $\alpha \subseteq \Omega$ and $H \leq S_{\Omega-\alpha}$ be transitive on α. Assume $\Gamma \in \mathcal{P}(H)$ and $\gamma \in \Gamma$ with $\alpha \cap \gamma \neq \varnothing$. Then either $\alpha \subseteq \gamma$, or $\gamma \subseteq \alpha$.*

Proof. Let $\omega \in \alpha \cap \gamma$. By hypothesis, $\alpha = \omega H$. Thus if H acts on γ, then $\alpha \subseteq \gamma$, so we may assume $h \in H - N_H(\gamma)$. Then $\gamma \cup \gamma h \subseteq \mathrm{Mov}(h) \subseteq \alpha$, completing the proof. □

1.2. *Let $P, Q \in \mathcal{P}(\Omega)$ with $P \vee Q = \infty$. Then $S_P \cap S_Q = 1$.*

Proof. As $P \vee Q = \infty$, $|\alpha \cap \beta| \leq 1$ for all $\alpha \in P$ and $\beta \in Q$. Let $\omega \in \Omega$. Then $\omega \in \alpha \in P$ and $\omega \in \beta \in Q$, so $\alpha \cap \beta = \{\omega\}$. Then as $H := S_P \cap S_Q$ acts on α and β, H fixes ω. As this holds for each $\omega \in \Omega$, $H = 1$. □

Given a lattice Λ and $x \in \Lambda$, set $\Lambda(\leq x) = \{y \in \Lambda : y \leq x\}$.

1.3. *Let Λ be a lattice isomorphic to $\Delta(m)$ for some m, and Ξ a sublattice of Λ containing $\Lambda(\leq x)$ for each $x \in \Xi - \{\infty\}$. Then $\Xi = \Lambda(\leq z)$ for some $z \in \Xi$, so $\Xi \cong \Delta(d)$ for some $0 \leq d \leq m$.*

Proof. Let $\mathcal{A}$ be the set of atoms of Λ, and for $x \in \Lambda$ and $\mathcal{B} \subseteq \mathcal{A}$, set

$$\mathcal{A}(x) := \mathcal{A} \cap \Lambda(\leq x) \quad \text{and} \quad y(\mathcal{B}) := \bigvee_{b \in \mathcal{B}} b.$$

Set $\mathcal{E} := \mathcal{A} \cap \Xi$ and $z := y(\mathcal{E})$. As Ξ is a sublattice of Λ, $z \in \Xi$, so by hypothesis either $z = \infty$ or $\Lambda(\leq z) \subseteq \Xi$. In the first case as $\Lambda \cong \Delta(m)$, $\mathcal{E} = \mathcal{A}$, and for each $x \in \Lambda$, $x = y(\mathcal{A}(x)) \in \Xi$, so $\Xi = \Lambda = \Lambda(\leq z)$. Thus we may assume $z \neq \infty$. Now for $x \in \Xi$, $\mathcal{A}(x) = \Lambda(\leq x) \cap \mathcal{A} \subseteq \mathcal{E}$, so $x = y(\mathcal{A}(x)) \leq y(\mathcal{E}) = z$. Thus again $\Xi = \Lambda(\leq z)$. □

1.4. *Assume $n := |\Omega| \equiv 2 \mod 4$ and $n > 2$. Then each primitive subgroup G of S is almost simple.*

Proof. Assume G is not almost simple. Then (cf. 2.5 in [4]) G stabilizes an affine, diagonal, or product structure on Ω.

In the first case as n is even, $n = 2^a$ for some positive integer a. As $n > 2$, $a > 1$, so $n \equiv 0 \mod 4$, a contradiction.

In the second case, $n = |L|^s$ for some nonabelian finite simple group L and positive integer s. However $|L| \equiv 0 \mod 4$, so $n \equiv 0 \mod 4$, again a contradiction.

Finally in the third case, $n = m^r$ for some integers $m \geq 5$ and $r > 1$. As n is even, so is m, so as $r > 1$, $n \equiv 0 \mod 4$, once again a contradiction. □

1.5. *Assume G is an almost simple transitive subgroup of S and $n := |\Omega|$ is a power of some prime p. Set $L := F^*(G)$. Then either*

(1) *L is transitive on Ω, or*

(2) *$p = 2 = |G : L|$, L has two orbits Ω_i, $i = 1, 2$, on Ω, and there is an involution in $C_A(L)$ interchanging Ω_1 and Ω_2.*

Proof. Since G is transitive on Ω, it follows that G is also transitive on the set $\mathcal{O} := \{\Omega_i : 1 \leq i \leq m\}$ of orbits of L on Ω. Set $k := |\Omega_1|$. Then $km = n = p^e$, so $k = p^a$ and $m = p^b$ with $a + b = e$. Further m divides $|G : L|$, and hence also $|\mathrm{Out}(L)|$. As L is transitive on Ω_1 and $k = p^a$, it follows (cf. 3.1 in [4]) that either

(i) p does not divide $|\mathrm{Out}(L)|$, or

(ii) $p = 2 = |\mathrm{Out}(L)|$, and either $L \cong A_k$, or $L \cong L_2(q)$ with q a Mersenne prime and $k = q + 1$.

In case (i), as $p^b = m$ divides $|\mathrm{Out}(L)|$, it follows that $m = 1$, so that (1) holds. Similarly in case (ii), either (1) holds or $G = \mathrm{Aut}(L)$ and $p = m = 2$, and we may assume the latter. Then for $\omega \in \Omega$ we have $L_\omega^G = L_\omega^L$, so there exists an equivalence $\alpha : \Omega_1 \to \Omega_2$ of the representations of L on Ω_i. Define $t \in S$ by $\omega t = \omega\alpha$ for $\omega \in \Omega_1$, and $\omega t = \omega\alpha^{-1}$ for $\omega \in \Omega_2$. As α is an equivalence of L-representations, $t \in C_S(L)$. By construction, t is an involution interchanging Ω_1 and Ω_2, so t has k cycles of length 2. Then as k is even, $t \in A$, so (2) holds. □

1.6. *Assume G is a transitive subgroup of S, $n := |\Omega| > 3$, and $\Sigma = \{\alpha_1, \alpha_2\}$ is a 2-subset of Ω such that $H := N_G(\Sigma)$ is transitive on $\Omega - \Sigma$. Assume Σ^G is not a partition of Ω.*

(1) *Either*

(i) *G is $5/2$-transitive on Ω, that is G is 2-transitive on Ω, and either G is 3-transitive or G_Σ has two orbits on $\Omega - \Sigma$ of equal length, or*

(ii) *H_{α_1} has two orbits θ_i, $i = 1, 2$, of equal length on $\Omega - \Sigma$ interchanged by H, and setting $\Omega_i := \theta_i \cup \{\alpha_i\}$, $\{\Omega_1, \Omega_2\}$ is a G-invariant partition of Ω such that $N_G(\Omega_i)$ is 2-transitive on Ω_i for $i = 1, 2$.*

(2) *If G is 2-transitive and affine on Ω, and G is not solvable, then*

$$D := F^*(G) \cong E_{2^e},$$

$G_{\alpha_1} \leq \mathrm{GL}(D)$*, G is 3-transitive on Ω, and either* $G_{\alpha_1} = \mathrm{GL}(D) \cong L_e(2)$*, or $e = 4$ and $G_{\alpha_1} \cong A_7$.*

(3) *Assume G is imprimitive and $1 \neq W \trianglelefteq G$ with G/W almost simple. Then case* (ii) *of* (1) *holds, $N_G(\Omega_i)^{\Omega_i}$ is affine, and $F^*(N_G(\Omega_i)^{\Omega_i}) \leq W^{\Omega_i}$.*

Proof. We first prove (1). Set $\alpha := \alpha_1$. As Σ^G is not a partition of Ω, G_α does not act on Σ by 5.18.2 in [1]. As $H = N_G(\Sigma)$ is transitive on $\theta = \Omega - \Sigma$ and $|\Sigma| = 2$, $H_\alpha = G_\Sigma$ and either H_α is transitive on θ, or H_α has two orbits θ_i, $i = 1, 2$, on θ of equal length m, interchanged by H. In the first case as G_α does not act on Σ, G_α is transitive on $\Lambda = \Omega - \{\alpha\}$, so G is 3-transitive and (1i) holds. Thus we may assume the second case holds.

Next if G_α is transitive on Λ, then G is $5/2$-transitive, and again (1i) holds, so we may assume otherwise. Thus we may assume G_α has orbits $\Delta = \Delta(\alpha) = \theta_1$ and $\Gamma = \{\alpha_2\} \cup \theta_2$ on Ω. Therefore G is a rank 3 group on Ω with parameters $k = m$ and $l = m+1$. By 16.3.2 in [1], there are nonnegative integers μ and $\lambda < k$ such that $\mu l = k(k-\lambda-1)$. Then as $(l, k) = 1$, it follows that l divides $k-\lambda-1$, so as $l > k > \lambda$, we conclude that $k = \lambda + 1$ and $\mu = 0$. Therefore by 16.4 in [1], G preserves the partition $\Xi := \{\omega^\perp : \omega \in \Omega\}$, where $\omega^\perp = \{\omega\} \cup \Delta(\omega)$. Then as $k = m$ and $l = m + 1$, we get $\Xi = \{\Omega_1, \Omega_2\}$. Further as G_α is transitive on Ω_2 and $\Omega_1 - \{\alpha\}$, $N_G(\Omega_1)$ is 2-transitive on Ω_1, so (1ii) holds, and the proof of (1) is complete.

Next assume the hypothesis of (2), and let $D = F^*(G)$. Then $D \cong E_{p^e}$ for some prime p, and $K = G_\alpha \leq \mathrm{GL}(D)$ is $3/2$-transitive on $D^\#$. Then $G_\Sigma = C_K(d)$ for $d \in D^\#$ with $\alpha d = \alpha_2$, and $C_K(d)$ is $1/2$-transitive on $D - \{1, d\}$, so as $C_K(d)$ acts on $\langle d \rangle$ and $D \neq \langle d \rangle$ since G is not solvable, it follows that $p = 2$. Now 4.8 in [2] contains a list of subgroups of $\mathrm{GL}(D)$ transitive on $D^\#$ (the classification of such subgroups is due to Hering and Liebeck; see the proof of 4.8 in [2] for discussion and references). As G_α is $3/2$-transitive and not solvable, we conclude from that list that (2) holds.

Finally assume the hypothesis of (3). Now $X := N_G(\Omega_i)$ is of index 2 in G, so X/U is almost simple, where $U := W \cap X \trianglelefteq G$. Next $X \leq N_S(\Omega_1) = S_1 \times S_2$, where $S_i = S_{\Omega_{3-i}}$ acts as the symmetric group of Ω_i. Let $\pi_i : X \to S_i$ be the ith projection, and $U_i := U\pi_i$. As H is transitive on Ξ, we get $U_1 \cong U_2$, so in particular, $U_i \neq 1$. As $Y_i := X^{\Omega_i}$ is 2-transitive, Y_i is almost simple or affine, so in particular Y_i has a unique minimal normal subgroup D_i. Thus as $U_i \neq 1$, $D_i \leq U_i$, so if Y_i is affine, then (3) holds. Suppose Y_i is almost simple and set $L := X^\infty$. Then $L\pi_i = D_i \leq U_i$ and hence $1 \neq U_i^\infty$ and $L = U^\infty X_{\Omega_i}^\infty$. As X/U is almost simple, it follows that $X_{\Omega_i}^{\infty} = D_{3-i}$ and $L = D_1 \times D_2$. Then as $U_i^{\infty} \neq 1$ and $U \trianglelefteq X$, it follows that $L \leq U$, and hence X/U is solvable, a contradiction. □

1.7. *Assume G is a transitive subgroup of S, Δ is a proper subset of Ω, $N_G(\Delta)$ is transitive on $\Omega - \Delta$, and $1 \neq Y \trianglelefteq N_G(\Delta)$ with $\Omega - \Delta \subseteq \mathrm{Fix}(Y)$. Then the following hold:*

(1) *If $N_G(\Delta)$ is 2-transitive on Δ, then Y is transitive on Δ.*

(2) *If $N_G(\Delta)$ is 2-transitive on Δ and $\Gamma \in \mathcal{P}'(G)$, then $\Gamma = \Delta^G$.*

(3) *If G is primitive on Ω, then G is 2-transitive on Ω and G is almost simple or affine.*

(4) *If G is primitive and affine on Ω, then $G \neq F^*(G)N_G(\Delta)$.*

Proof. Let $H := N_G(\Delta)$ and $\theta := \Omega - \Delta$. We first prove (1) and (2), so we assume H is 2-transitive on Δ. Then H^{Δ} is primitive. As $\theta \subseteq \mathrm{Fix}(Y)$, Y is faithful on Δ, so $1 \neq Y^{\Delta} \trianglelefteq H^{\Delta}$ and hence (1) holds.

Suppose $\Gamma \in \mathcal{P}'(G)$ and let $\gamma \in \Gamma$ with $\gamma \cap \Delta \neq 1$. By (1) and 1.1, either $\Delta \subseteq \gamma$ or $\gamma \subseteq \Delta$. In the former case $\Delta = \gamma$ as H is transitive on θ, and in the latter $\gamma = \Delta$ as H is 2-transitive on Δ. Thus (2) holds.

We next prove (3) and (4), so we assume G is primitive on Ω. Then statement (3) follows from 15.17 in [1]. Finally assume G is affine on Ω and set $D = F^*(G)$. As $\theta = \mathrm{Fix}(Y)$, it follows that $C_D(Y)$ is regular on θ, so $1 \neq C_D(Y) < D$ is H-invariant. But if $G = DH$, then as G is primitive on Ω, H is irreducible on D, contradicting $C_D(Y)$ H-invariant. □

1.8. *Let $n := |\Omega|$.*

(1) $m_2(S) = [n/2]$.

(2) *If H is a primitive subgroup of S with $m_2(H) \geq (n-2)/4$, then H is almost simple or affine.*

Proof. Let $n_0 := [n/2]$, E the subgroup of a Sylow 2-subgroup T of S generated by the transpositions in T, and $M := N_S(E)$. Then $T \leq M$ so $m_2(M) = m_2(S)$. Further $m_2(E) = n_0$ and M is the split extension of E by S_{n_0} with E the permutation module for M/E. Hence by B.3.2.4 and B.2.4 in [9], $m_2(M) = m_2(E)$, establishing (1).

Assume H satisfies the hypotheses of (2) but H is neither affine nor almost simple. Then, replacing H by a larger primitive subgroup of S if necessary, and appealing to 2.2 in [4], we may assume H is the stabilizer of a product structure or diagonal structure on Ω. In Case I it follows from 2.2 in [4] that H is a wreath product of S_m by S_k for some $m \geq 5$ and $k \geq 2$, so H has a normal subgroup

$D = D_1 \times \cdots \times D_k$ with $D_i \cong S_m$ for $i \in I = \{1, \ldots, k\}$. In this case set $X = H$. In Case II we have $F^*(H) = D = D_1 \times \cdots \times D_k$ with $D_i \cong L$ for some nonabelian simple group L, $k \geq 2$, and $H/D = X/D \times Y/D$, where $X/D = \mathrm{Sym}(I)$ and $Y/D \cong \mathrm{Out}(L)$. In each case X acts on the D_i as $\mathrm{Sym}(I)$ with kernel D via $D_i^x = D_{ix}$ for $x \in X$ and $i \in I$.

Let A be an elementary abelian 2-subgroup of X of rank $m_2(X)$ such that $B := A \cap D$ is of maximal order. Claim $A = B$. Assume otherwise. Then A moves some $i \in I$. Set $J = iA$, let F be a complement to A_i in A, and for $j \in J$ let E_j be the projection of B on D_j. As $m_2(A) = m_2(X)$, it follows that $E_j \neq 1$. Set $E = \langle E_j : j \in J \rangle$. Now F is regular on J, so $m_2(F) \leq m_2(E/C_E(F))$, and hence, as in the proof of (1), $m_2(A_i E) \geq m_2(A)$, contradicting the maximality of B. This establishes the claim.

By the claim, we have $m_2(X) = m_2(D)$, so $m_2(X) = kr$ where $r := m_2(D_i)$. In Case I, $H = X$, so $m_2(H) = kr$ and $r = [m/2]$ by (1). In Case II we have $H/X \cong \mathrm{Out}(L)$, so $m_2(H) \leq kr + 3$ as $m_2(\mathrm{Out}(L)) \leq 3$ (cf. 2.5.12 in [11] when L is of Lie type, 5.2.1 in [11] when L is an alternating group, and Chapter 5 in [11] when L is sporadic). Further $r = m_2(L)$. Thus it remains to show

$$(n-2)/4 > kr + e \tag{$*$}$$

where $e = 0$ in Case I and $e = 3$ in Case II. Assume otherwise.

In Case I, $n = m^k$ by 2.5 in [4], while $r = [m/2] \leq m/2$ by (1). Then we get $(m^k - 2)/4 \leq km/2$, so $m^k \leq 2(km + 1)$, a contradiction as $m \geq 5$ and $k \geq 2$.

Therefore Case II holds. Hence by 2.5 in [4], $n = m^{k-1}$ where $m := |L|$. Observe $m = 2^r s$ where $s \geq 15$ as m is divisible by distinct odd primes. Now we find that $(m^{k-1} - 2)/4 \leq kr + 3$, so

$$(15 \cdot 2^r)^{k-1} \leq m^{k-1} \leq 4kr + 14,$$

again a contradiction as $k \geq 2 \leq r$. □

1.9. *Assume G is primitive on Ω but not almost simple, and $n := |\Omega| \geq 10$ is even. Let Σ be a $(2, n/2)$-partition of Ω, $T = A_\Sigma$, and z the fixed-point-free involution in S whose cycles are the clock of Σ. Assume $z \in X := T \cap G$ and*

$$m_2(X) \geq (n-2)/4.$$

Then $n = 16$, G is the stabilizer in S of an affine structure on Ω, $m_2(X) = 4$, and $G = \langle X^G \rangle$.

Proof. Set $Z := \langle z \rangle$. By 1.8, G is affine on Ω, so $F^*(G) = E \cong E_{2^e}$ for some $e \geq 4$ as $n \geq 10$ is even. Thus

$$m_2(X) \geq (n-2)/4 = (2^{e-1} - 1)/2.$$

Set $G^* := G/E$.

As z is the unique fixed-point-free involution in X, $X \cap E \leq Z$. Claim $z \in E$. Assume otherwise and let $F := C_E(z)$ and $f := m_2(F)$. Then $f \geq e/2 \geq 2$. Further $[F, X] \leq E \cap X = 1$. But F has $2^e/2^f = 2^{e-f}$ regular orbits on Ω, so each member of the set θ of orbits of F on Σ is of length at least 2^{f-1}. Thus we have $|\theta| \leq 2^{e-1}/2^{f-1} = 2^{e-f}$, so

$$(2^{e-1} - 1)/2 \leq m_2(X) \leq m_2(C_{S_\Sigma}(F)) = |\theta| = 2^{e-f}.$$

We conclude that $f \leq 2$, so as $f \geq e/2 \geq 2$ it follows that $e = 4$ and $f = 2$. Thus $F = [E, z]$ so $zF \subseteq z^E$, and hence ZF is semiregular on Ω. But then each member of θ is of order 4, so $|\theta| = 2$ and $4 \leq m_2(X) = 2$, a contradiction. This completes the proof of the claim.

By the claim, $E \cap X = Z$, so $E \leq C_G(Z) \leq N_G(X)$. Thus

$$m_2(X^*) = m_2(X) - 1 \geq (2^{e-1} - 3)/2.$$

Further $[E, X] \leq Z$, so X^* induces a group of transvections on E with center Z. Hence $m_2(X^*) \leq e - 1$. Thus $(2^{e-1} - 3)/2 \leq e - 1$, so as $e \geq 4$ we conclude that $e = 4$ and $m_2(X^*) = 3$.

Now as G is primitive on Ω, G^* is irreducible on E. But as $m_2(X^*) = 3$, X^* is the radical of the parabolic of $\mathrm{GL}(E)$ stabilizing Z, so $G^* = \langle X^{*G^*} \rangle = \mathrm{GL}(E)$. Therefore $G^* = \langle X^G \rangle^*$, so $Y = \langle X^G \rangle$ is irreducible on E. Hence we obtain that $E = \langle E^Y \rangle \leq Y$, so $G = EY = Y$, completing the proof. □

2 Lower signalizer lattices

In this section we assume:

Hypothesis 2.1. G is a finite group, L is a nonabelian finite simple group, $\gamma = (G, H, J) \in \mathcal{T}(L)$, and G_0 is a normal subgroup of G.

Notation 2.2. Set $\hat{G} := G_0 J$, write H_0 for the preimage in H of $F^*(H/J)$, and let

$$\mathcal{W}_0 := \{W \in \mathcal{W}(\gamma, G_0) : W \leq \hat{G}\}.$$

Partially order $\mathcal{W}_0$ by inclusion, and let $\Xi = \Xi(\gamma, G_0)$ be the poset obtained by adjoining a greatest member ∞ to the poset $\mathcal{W}_0$. Observe J is the least element of Ξ; we sometimes write 0 for J regarded as a member of Ξ. Set $\mathcal{W}_0' := \mathcal{W}_0 - \{0\}$, and for $W \in \mathcal{W}_0'$ let

$$\mathcal{W}_0(> W) := \{U \in \mathcal{W}_0 : W < U\} \quad \text{and} \quad \mathcal{W}_0(< W) := \{U \in \mathcal{W}_0' : U < W\}.$$

Define $\mathcal{W}_0(\geq W)$ and $\mathcal{W}_0(\leq W)$ similarly. For $X \in \mathcal{I}_G(J)$, set $\bar{X} := XJ$.

Remark 2.3. We call a lattice of the form $\Xi(\tau, X_0)$, for some $\tau = (X, H_X, J_X) \in \mathcal{T}(L)$ and $X_0 \trianglelefteq X$, a *lower signalizer lattice*. From (the proof of) 2.11.2 in [6], $\Xi(\tau)$ is isomorphic to the dual of the poset $\Delta(\tau, X_0)$ defined in Section 2 of [6]. Thus we can appeal to results from [6] on such posets.

2.4. (1) $\Xi := \Xi(\gamma, G_0)$ *is a lattice.*

(2) *Suppose* $\infty \neq W_i \in \Xi$ *for* $i = 1, 2$. *Then*

$$W_1 \wedge W_2 = W_1 \cap W_2,$$

and if $W_1 \vee W_2 \neq \infty$, *then* $W_1 \vee W_2 = \langle W_1, W_2 \rangle$.

Proof. The lemma follows from (the proof of) parts (2) and (3) of 2.11 in [6], which use the isomorphism of Remark 2.3. □

2.5. *Let* $H \leq M \leq G$. *Then*

(1) $\tau_M := (M, H, J) \in \mathcal{T}(L)$.

(2) $\Xi_M := \Xi(\tau_M, G_0 \cap M)$ *is a lower signalizer lattice, and a sublattice of* Ξ.

(3) *For* $W \in \Xi_M$, $\mathcal{W}_0(\leq W) \subseteq \Xi_M$.

(4) *The inclusion map is an isomorphism of*

$$\Xi(\tau_{\langle \mathcal{W}_0(\gamma, G_0), H\rangle}, \langle \mathcal{W}_0(\gamma, G_0), H\rangle \cap G_0)$$

with Ξ.

Proof. The proof is straightforward. □

2.6. $J = 1$ *if and only if* H *is almost simple.*

Proof. As $\gamma \in \mathcal{T}(L)$, H/J is almost simple. Thus if $J = 1$, then H is almost simple. Conversely if H is almost simple, then $F^*(H)$ is the unique minimal normal subgroup of H, so if $J \neq 1$, then $F^*(H) \leq J$. This is impossible as $H/F^*(H)$ is solvable, whereas H/J is not. □

2.7. *Assume* $V \in \mathcal{W}_0$. *Then:*

(1) *For each* $W \in \mathcal{V}_V(H)$, $W \in \mathcal{W}_0$.

(2) *If* $F^*(HV) \leq V$, *then* $F^*(HV)J \in \mathcal{W}_0$.

Proof. Part (1) follows from 2.5.1 in [6]. Assume $F^*(HV) \leq V$. Then we obtain $J \leq F^*(HV)J \in \mathcal{I}_V(H)$, so (1) implies (2). □

2.8. *Assume* $X_i \in \mathcal{I}_{\hat{G}}(H)$, $1 \leq i \leq 3$, *with* $X_1 \trianglelefteq X_2 \leq X_3$, $H \cap X_3 \leq X_2$, $\bar{X}_1 \in \mathcal{W}$, *and* X_2/X_1 *solvable. Then* $\bar{X}_i \in \mathcal{W}_0$ *for* $1 \leq i \leq 3$.

Proof. If $\bar{X}_2 \in \mathcal{W}$, then as $H \cap X_3 \leq X_2$, we have $H \cap X_3 = H \cap X_2 \leq J$, so $\bar{X}_3 \in \mathcal{W}$. Then as $X_3 \leq \hat{G}$, the lemma holds. So assume $\bar{X}_2 \notin \mathcal{W}$. Applying 3.6.2 in [6] to $\bar{X}_1, X_2$ in the role of V, W, we conclude that $P := N_{X_2}(\bar{X}_1)$ is H-invariant with $\bar{X}_1 \cap X_2 \leq P$ and $L \cong P/(\bar{X}_1 \cap X_2)$. But $\bar{X}_1 \cap X_2 = X_1(J \cap X_2)$, and P/X_1 is a subgroup of the solvable group X_2/X_1, so the image

$$L \cong P/(X_1(J \cap X_2))$$

of P/X_1 is solvable, a contradiction. □

2.9. *Assume $V \in \mathcal{W}_0$ and $M \in \mathcal{O}_G(HV)'$ with*

- $F^*(HV) \leq F^*(M) \leq \hat{G}$,
- $F^*(HV) \leq V$, *and*
- $N_{F^*(M)}(F^*(HV))/F^*(HV)$ *is solvable.*

Then $F^*(M)J$ and $N_{F^*(M)}(F^*(HV))J$ are in $\mathcal{W}_0$.

Proof. Let $X_1 = F^*(HV)$, $X_3 = F^*(M)$, and $X_2 = N_{X_3}(X_1)$. By 2.7.2, we have $\bar{X}_1 \in \mathcal{W}_0$, and by hypothesis X_2/X_1 is solvable. As H acts on X_1, it follows that $H \cap X_3 \leq N_{X_3}(X_1) = X_2$. Thus the lemma follows from 2.8. □

2.10. *Assume for all $V \in \mathcal{W}_0$ that:*

(a) $F^*(HV) \leq V$.

(b) *There exists $M(V) \in \mathcal{M}(HV)$ such that*

$$F^*(HV) \leq F^*(M(V)) \leq \hat{G}$$

and $N_{F^*(M(V))}(F^*(HV))/F^*(HV)$ is solvable.

Then the following hold:

(1) *$F^*(HV)J$ and $F^*(M(V))J$ are in $\mathcal{W}_0$.*

(2) *Let $\mathcal{X}$ consist of the subgroups X of $F^*(H)$ such that $X \trianglelefteq H$ and*

$$C_{F^*(H)}(X) \leq X.$$

Assume in addition that:

(c) *For each $X \in \mathcal{X}$, $N_G(X) \leq M(J)$, and for all $V \in \mathcal{W}_0$,*

$$F^*(M(J)) \leq F^*(M(V)).$$

Then either Ξ is connected or $F^*(M(J)) \leq J$.

(3) *Assume the hypothesis of* (2) *and assume* Ξ *is disconnected and for each* $V \in \mathcal{W}_0$, $K(V)$ *is a normal subgroup of* $M(V)$ *containing* $F^*(M(V))$ *such that* $K(V)/F^*(M(V))$ *is solvable,* $K(V) \le \hat{G}$, *and* $K(J) \le K(V)$. *Then we have* $K(V)J \in \mathcal{W}_0$ *and* $K(J) \le J$.

Proof. Part (1) follows from 2.7.2 and 2.9, so assume the hypothesis of (2). Let $P := F^*(M(J))$, let V be a minimal member of $\mathcal{W}_0'$, and set $Q := F^*(M(V))$ and $R := F^*(HV)$. We may assume $P \not\le J$, and it remains to show Ξ is connected. For $U \in \mathcal{W}_0'$, let $\mathcal{C}(U)$ be the connected component of Ξ' containing U. As $P \not\le J$, $\bar{P} \in \mathcal{W}_0'$ by (1). By (c), $P \le Q$, and $\bar{Q} \in \mathcal{W}_0$ by (1), so $\bar{Q} \in \mathcal{W}_0'$ and $\mathcal{C}(\bar{P}) = \mathcal{C}(\bar{Q})$. Similarly $\bar{R} \in \mathcal{W}_0$ by (1), and $\bar{Q} \ge \bar{R} \le V$ by (a) and (b), so if $R \not\le J$, then $\bar{R} \in \mathcal{W}_0'$ and $\mathcal{C}(V) = \mathcal{C}(\bar{Q}) = \mathcal{C}(\bar{P})$. Thus Ξ is connected unless we can choose V with $\mathcal{C}(V) \ne \mathcal{C}(\bar{P})$ and $R \le J$, and we may assume the latter.

As $R \le J$, $R \in \mathcal{X}$. Also $R \le S = F^*(H) = F^*(HJ) \le F^*(M(J)) = P$ by (b), and $HV \le N_G(R) \le M(J)$ by (c). Then V acts on P so $\langle V, \bar{P}\rangle = VP$. Hence as $\mathcal{C}(V) \ne \mathcal{C}(\bar{P})$, $VP \notin \mathcal{W}_0$ and $V \cap \bar{P} = J$ by 3.8 in [6]. Next by 3.9.3.c in [6], $U := NV(\bar{P}) \in \mathcal{W}_0'$, so $U = V$ by minimality of V. Then by 3.9.3.d in [6], $\theta(V) \le N_{\bar{P}}(V)$, where $\theta(V)$ is defined in 3.2 of [6]. Therefore as $V \cap \bar{P} = J$, $\theta(V) \le N_{\bar{P}}(J)$. As $S := F^*(H) \le J$ by (a), $S = F^*(J)$, so $\theta(V) \le N_{\bar{P}}(S)$. But $S \le J \cap P$, so $N_{\bar{P}}(S)/J = N_P(S)J/J \cong N_P(S)/(J \cap P)$ is an image of $N_P(S)/S$, which is solvable by (b), whereas $\theta(V)/J \cong L$ by 3.2.4 in [6], a contradiction. This completes the proof of (2).

Assume the hypothesis of (3). By (1), $\bar{Q} \in \mathcal{W}_0$, so as $Q' = K(V) \trianglelefteq M(V)$ and Q'/Q is solvable, $\bar{Q}' \in \mathcal{W}_0$ by 2.8 applied with $X_1 = Q$ and $X_2 = X_3 = Q'$. Assume $P' = K(J) \not\le J$. As Ξ is disconnected, we can choose V with $\mathcal{C}(V) \ne \mathcal{C}(\bar{P}')$. By hypothesis, $\bar{P}' \le \bar{Q}'$, so $\mathcal{C}(\bar{P}') = \mathcal{C}(\bar{Q}')$. Further $\bar{Q} \le \bar{Q}'$ and from the proof of (2), $\bar{Q} \ge \bar{R} \le V$, so if $R \not\le J$, then

$$\mathcal{C}(V) = \mathcal{C}(\bar{Q}) = \mathcal{C}(\bar{Q}') = \mathcal{C}(\bar{P}'),$$

contrary to the choice of V. Thus $R \le J$, and then the argument in the last paragraph of the proof of (2) applied to P' in place of P supplies a contradiction. □

2.11. *Let* $\mathcal{H}$ *be the set of* H*-invariant subgroups* X *of* H_0 *such that* $H_0 = XJ$. *Then* $\mathcal{H}$ *has a least element* H_*.

Proof. Suppose $H_i \in \mathcal{H}$ for $i = 1, 2$ and let $J_i := J \cap H_i$, $H_3 := [H_1, H_2]$, and $H^* := H/J_1$. As $H_i \trianglelefteq H$ for $i = 1, 2$, we get $H_3 \le H_1 \cap H_2$, so it suffices to show that $H_3 \in \mathcal{H}$.

Now $H_1^* \cong H_1/J_1 \cong H_1J/J = H_0/J \cong L$ is nonabelian simple and normal in H^*. As $H_0 = H_1J$, it follows that $H_0^* = H_1^*J^*$ and $H_1^* \cap J^* = J_1^* = 1$, so

$H_0^* = H_1^* \times J^*$. As $H_0^* = H_2^* J^*$ and H_1^* is nonabelian simple, we get

$$H_1^* = [H_1^*, H_2^*] = [H_1, H_2]^* = H_3^*,$$

so $H_1 = H_3 J_1$. Thus $H_0 = H_1 J = H_3 J$, so $H_3 \in \mathcal{H}$, completing the proof. □

2.12. *Let $X \in \mathcal{I}_{\hat{G}}(H)$. Then either $\bar{X} \in \mathcal{W}_0$ or $H_* \leq X$.*

Proof. If $\bar{X} \in \mathcal{W}$, then as $\bar{X} \leq \hat{G}$, we have $\bar{X} \in \mathcal{W}_0$. So assume $\bar{X} \notin \mathcal{W}$. Then by 3.5 in [6], $H_0 = (X \cap H_0)J$. Thus $X \cap H_0$ is in the set $\mathcal{H}$ of 2.11, so $H_* \leq X$ by 2.11. □

2.13. *Let $W \in \mathcal{W}_0$, $X \in \mathcal{I}_{\hat{G}}(HW)$, and $Y \leq X \cap W$ with $Y \trianglelefteq X$ and X/Y solvable. Then $WX \in \mathcal{W}_0$.*

Proof. Assume $WX \notin \mathcal{W}_0$. As $X \leq \hat{G}$, also $WX \leq \hat{G}$, so if $WX \in \mathcal{W}$, then also $WX \in \mathcal{W}_0$. Thus we have $WX \notin \mathcal{W}$. Then by 3.6.2 in [6], $V := N_X(W)$ satisfies $Z = W \cap X \leq V$ and $L \cong V/Z$. Let $X^* = X/Y$, so that X^* is solvable. As $Y \leq Z \leq V$, it follows that $L \cong V/Z \cong V^*/Z^*$ is a section of the solvable group X^*, a contradiction. □

2.14. *Assume $W \in \mathcal{W}_0$ and WH is almost simple. Then $W = 1$.*

Proof. If not, then as $W \trianglelefteq WH$, $F^*(WH) \leq W$. Thus WH/W is solvable, contradicting WH/W almost simple. □

2.15. *Assume $U \in \mathcal{W}_0'$ and $Y := UH$ is represented as a 2-transitive group of permutations on a set Γ. Set $K := Y_\Gamma$ and let D be the preimage in Y of $F^*(Y^\Gamma)$. Assume either $K < U$ or $K \leq J$. Then*

(1) *Y^Γ is affine.*

(2) *$D^\Gamma \leq U^\Gamma$.*

(3) *$D \leq U$ and $\bar{D} \in \mathcal{W}_0$.*

Proof. As Y^Γ is 2-transitive, Y^Γ is almost simple or affine. In particular $F^*(Y^\Gamma)$ is the unique minimal normal subgroup of Y^Γ. By hypothesis either $K < U$ or $K \leq J$. In the latter case as $U \in \mathcal{W}_0'$, again $K < U$. Thus $1 \neq U^\Gamma \trianglelefteq Y^\Gamma$, so (2) holds.

Next if Y^Γ is almost simple, then Y^Γ/U^Γ is solvable by (2), so as $K \leq U$, also UH/U is solvable, a contradiction. Therefore (1) holds.

As $K \leq U$, (2) says that $D \leq U$. Thus $\bar{D} \in \mathcal{W}_0$ by 2.7.1, so (3) holds. □

2.16. *Let $V \in \mathcal{W}_0$ and set $\alpha := (G, HV, V)$. Then*

(1) *$\alpha \in \mathcal{T}(L)$ and G, α, G_0 satisfies Hypothesis* 2.1.

(2) *$\mathcal{W}_0(\alpha) = \{W \in \mathcal{W}_0 : V \leq W\}$.*

Proof. Straightforward. □

3 $D\Delta(m_1, \ldots, m_t)$-lower signalizer lattices

In this section we assume:

Hypothesis 3.1. G is an almost simple finite group, L is a nonabelian finite simple group, $\gamma := (G, H, J) \in \mathcal{T}(L)$, $\Lambda := \Xi(\gamma)$ is a $D\Delta(m_1, \ldots, m_t)$-lattice for some integers $t > 1$, $m_i > 2$, and $G = \langle \mathcal{W}_0, H\rangle$. For $X \in \mathcal{I}_G(J)$, set $\bar{X} := XJ$.

Remark 3.2. Observe that Hypothesis 2.1 is satisfied with $G_0 = F^*(G)$. We adopt Notation 2.2. In addition let $\mathcal{W}_0^*$ be the set of minimal members of the poset $\mathcal{W}_0'$, and let $\mathcal{W}_0^!$ be the set of maximal members of $\mathcal{W}_0'$. Define H_* as in 2.11.

3.3. $G = \langle \mathcal{W}_0^*, H\rangle$.

Proof. As Λ is a $D\Delta(m_1, \ldots, m_t)$-lattice, for each $W \in \mathcal{W}_0$,

$$W = \langle U : U \in \mathcal{W}_0^* \text{ and } U \leq W\rangle$$

(cf. 3.4). Hence the lemma follows as $G = \langle \mathcal{W}_0, H\rangle$ by 3.1. □

3.4. *Let $\mathcal{C}$ be a connected component of Λ', $\Sigma := \mathcal{C} \cup \{0, \infty\}$, and $\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*$. For $\alpha \subseteq \mathcal{C}^*$, set $L_\alpha := \langle W : W \in \alpha\rangle$, with $L_{\mathcal{C}^*} := \infty$ and $L_\varnothing := J$. Let $\mathfrak{C}$ be the lattice of all subsets of $\mathcal{C}^*$. Then the following hold:*

(1) *$\Sigma \cong \Delta(m_i)$ for some $1 \leq i \leq t$.*

(2) *The map $\alpha \mapsto L_\alpha$ is an isomorphism of posets of $\mathfrak{C}$ with Σ.*

(3) *$L_{\alpha\cup\beta} = \langle L_\alpha, L_\beta\rangle$ and $L_{\alpha\cap\beta} = L_\alpha \cap L_\beta$.*

(4) *$\langle \mathcal{C}^*\rangle \notin \mathcal{W}_0$.*

Proof. Part (1) follows as Λ is a $D\Delta(m_1, \ldots, m_t)$-lattice. Then (1) and 2.4 imply (2), (2) implies (3), and (1) and (2) imply (4), since $\langle \mathcal{C}^*\rangle = L_{\mathcal{C}^*} = \infty$. □

3.5. *The following hold:*

(1) *Assume $U, V \in \mathcal{W}_0$ with $V < U$. Then $V = N_U(HV)$.*

(2) *$|U : V| \neq 2$.*

(3) *$|U : J| \neq 2$.*

Proof. If (2) fails, then $[H, U] \le V \le HV$, so (1) supplies a contradiction. Also (2) implies (3). Thus it suffices to prove (1).

Assume the setup of (1), with $V < N_U(HV)$. By 2.5.1 in [6], $N_U(HV) \in \mathcal{W}_0$, so we may take $U = N_U(HV)$. Form $\alpha := (G, HV, V)$ as in 2.16. By that lemma, $\Gamma_1 := \mathcal{W}_0(\ge V) \cup \{\infty\}$ is a sublattice of Λ. Let Γ_0 be the connected component of Γ_1' containing U, and $\Gamma := \Gamma_0 \cup \{0, \infty\}$. From 3.4, $\Gamma \cong \Delta(m)$ for some $m > 1$. Now U acts on HV and $V = HV \cap U$, so U is represented as a group of automorphisms of Γ via conjugation. Further $U \in \Gamma$. Let Γ^* be the minimal members of Γ, $\mu := \{W \in \Gamma^* : W \le U\}$, $\eta := \Gamma^* - \mu$, and $X := \langle \eta \rangle$. Then we find that $X \cap U = V$ and U acts on μ and Γ^*, and hence also on η and X. By 3.4.4, $UX \notin \mathcal{W}_0$, so by 3.9.3 in [6], setting $U_1 := N_U(X)$ and $X_1 := N_X(U)$, we obtain that $V = U_1 \cap X_1 \trianglelefteq U_1 X_1 = Y$, and setting $Y^* = Y/V$, $Y^* = X_1^* \times U_1^*$ with $H_*^* \cong L$ a full diagonal subgroup of Y^*. As H centralizes U^*, this is a contradiction. □

3.6. *The following hold:*

(1) *For $W \in \mathcal{W}_0$, $F^*(G) \not\le WH$, so $\mathcal{M}(WH) \neq \varnothing$.*

(2) $|\mathcal{M}(H)| > 1$.

Proof. Suppose (1) fails for some W. Then $W = 1$ by 2.14, so $F^*(G) \le H$. Thus for each $U \in \mathcal{W}_0$, $F^*(G) \le UH$, so $U = 1$, contrary to $\Lambda' \neq \varnothing$. This establishes statement (1). Then (1) and 3.3 imply (2). □

3.7. *Let $\mathcal{C}$ be a connected component of Λ', and adopt the notation of* 3.4. *Assume $Y \trianglelefteq X \le G$ with $H \le X$, and set $\mathcal{D} := \{d \in \mathcal{C}^* : L_d \le X\}$. For $\delta \subseteq \mathcal{D}$, set $Y_\delta := L_\delta \cap Y$. Set $\mathcal{B} := \{b \in \mathcal{D} : Y_b \le J\}$ and $\mathcal{A} := \mathcal{D} - \mathcal{B}$. Then for each $\alpha \subseteq \mathcal{A}$ and $\beta \subseteq \mathcal{B}$:*

(1) $L_\alpha = \bar{Y}_\alpha$.

(2) $Y_\alpha = \langle Y_a : a \in \alpha \rangle$.

(3) *The map $\alpha \mapsto Y_\alpha$ is an isomorphism of the lattice $\mathfrak{A}$ of all subsets of $\mathcal{A}$ with $\mathcal{V}_{Y_\mathcal{A}}(H)$.*

(4) *L_β acts on Y_α.*

(5) $Y_\beta \subseteq J$.

Proof. Let $a \in \alpha$, $b \in \beta$. By 2.7, $\bar{Y}_a \in \mathcal{W}_0$ and as $a \in \mathcal{A}$, $Y_a \not\le J$, so $\bar{Y}_a \in \mathcal{W}_0'$. Then as $\bar{Y}_a \le L_a \in \mathcal{W}_0^*$, $L_a = \bar{Y}_a$. By 3.4,

$$L_\alpha = \langle L_a : a \in \alpha \rangle = \langle \bar{Y}_a : a \in \alpha \rangle = Z_\alpha J,$$

where $Z_\alpha := \langle Y_a : a \in \alpha\rangle$. Thus $Z_\alpha \leq L_\alpha \cap Y = Y_\alpha$ and hence (1) holds. Further $Y_\alpha = L_\alpha \cap Y = Z_\alpha J \cap Y = Z_\alpha(Y \cap J) = Z_\alpha$ as $J \cap Y \leq Y_a \leq Z_\alpha$. Thus (2) holds. Then (1) and (2) imply the map $\varphi : \mathfrak{A} \to \mathcal{V}_{Y_\mathcal{Q}}(H)$ of (3) is an injective map of lattices. By 2.7, φ is a surjection, so (3) holds.

Next $Y_{a,b} = L_{a,b} \cap Y$ is H-invariant, so $\bar{Y}_{a,b} \in \mathcal{W}_0$ by 2.7. Further we have $Y_a \leq Y_{a,b} \leq L_{a,b}$, so $L_a = \bar{Y}_a \leq \bar{Y}_{a,b} \leq L_{a,b}$. Therefore by 3.4.2, $\bar{Y}_{a,b} = L_a$ or $L_{a,b}$. But in the latter case, $L_b = L_b \cap L_{a,b} = L_b \cap Y_{a,b}J \leq (L_b \cap Y)J = J$, a contradiction. Thus $\bar{Y}_{a,b} = L_a$, so $Y_{a,b} = Y_a$. Thus Y_a is L_b-invariant, so by 3.4.2, $L_\beta = \langle L_b : b \in \beta\rangle$ acts on $Y_\alpha = \langle Y_a : a \in \alpha\rangle$, establishing statement (4). Finally $\bar{Y}_\beta \leq L_\beta$, so $\bar{Y}_\beta = L_\delta$ for some $\delta \subseteq \beta$. But if $b \in \delta$, then

$$L_b = L_b \cap L_\delta \leq L_b \cap YJ = (L_b \cap Y)J = J,$$

a contradiction. Thus $L_\beta \cap Y \leq J$, establishing (5). □

3.8. *Assume $D \in \mathcal{I}_G(H)$ with $\bar{D} \in \mathcal{W}_0'$. Let $\mathcal{C}$ be the connected component of $\bar{D}$ in Λ', and adopt the notation of* 3.4. *Then $\bar{D} = L_\alpha$ for some $\varnothing \neq \alpha \subset \mathcal{C}^*$, and setting $\beta := \mathcal{C}^* - \alpha$, $W = L_\beta$ does not act on D.*

Proof. Assume otherwise, and let $X := WDH$ and $X^* := X/D$. By 3.4.4, we have $WD \notin \mathcal{W}_0$, so $H_* \leq WD$ by 2.12. Let V be the preimage in W of H_*^*. As H_* and W are H-invariant, V is H-invariant. Thus we get $\bar{V} \in \mathcal{W}_0$ by 2.7.1. Now $H_* \leq VD$ so $\bar{V}\bar{D} \notin \mathcal{W}_0$. Notice $\bar{V} \cap \bar{D} \leq W \cap \bar{D} = L_\beta \cap L_\alpha = J$ by 3.4.3. Set $V_1 := N_{\bar{V}}(\bar{D})$ and $D_1 := N_{\bar{D}}(\bar{V})$; then $J = V_1 \cap D_1$ and $V_1, D_1 \in \mathcal{W}_0$. Set $X_1 := V_1 D_1$ and $X_1^* := X/J$. By 3.9.3 in [6], $V_1^* \cong D_1^* \cong L$ and H_*^* is a full diagonal subgroup of X_1^*. Thus $V_1, D_1 \in \mathcal{W}_0^*$, so $V_1 = L_b$ and $D_1 = L_a$ for some $a \in \alpha$ and $b \in \beta$. Thus as $|\mathcal{C}| \geq 3$, it follows that $V_1 D_1 = L_{a,b} \in \mathcal{C}$, contradicting $H_* \leq V_1 D_1$. □

4 Semisimple subgroups

In this section we assume:

Hypothesis 4.1. Hypothesis 3.1 holds, $M \in \mathcal{O}_G(H)$, and $D \trianglelefteq M$ is the direct product of a set $\mathcal{L} = \{D_1, \ldots, D_m\}$ of subgroups permuted transitively by H, with $m > 1$ and $H_* \leq D$.

Notation 4.2. For $X \leq M$ and $E \in \mathcal{L}$, set $X_E := X \cap E$, $X_D := X \cap D$, and write $\tilde{X}_E$ for the projection of X_D on E. Set

$$\tilde{X} := \prod_{E \in \mathcal{L}} \tilde{X}_E.$$

4.3. *The following hold:*

(1) $\tau_M := (M, H, J) \in \mathcal{T}(L)$.

(2) $\Xi_M := (\tau_M, F^*(G) \cap M)$ *is a lower signalizer lattice, and a sublattice of* Λ.

(3) *For* $\infty \neq W \in \Xi_M$, $\mathcal{W}_0(\leq W) \subseteq \Xi_M$.

Proof. This is immediate from 2.5. □

4.4. *Let* $V \in \hat{\Xi}_M := \Xi_M - \{\infty\}$. *Then:*

(1) *For each* $E \in \mathcal{L}$, $\tilde{H}_{*E}\tilde{V}_E/\tilde{V}_E \cong L$.

(2) $\tilde{V}J \in \mathcal{W}_0$.

(3) $H_{*D}\tilde{V}/\tilde{V}$ *is a full diagonal subgroup of* $\tilde{H}_{*D}\tilde{V}/\tilde{V}$.

Proof. The proofs of 5.6 and 5.7 in [6] go through under our weaker hypothesis here. □

4.5. $\Xi_M = \Xi_1 * \cdots * \Xi_s$, *where for each* i, $\Xi_i \leq \Lambda_i$ *for some connected component* Λ'_i *of* Λ', *and either* $\Xi_i = \Lambda_i$, *or* $\Xi_i = \Lambda_i(\leq \gamma_i) \cup \{\infty\}$ *for some* $\gamma_i \in \Lambda'_i$.

Proof. By 4.3, Ξ_M is a sublattice of Λ containing $\mathcal{W}_0(\leq W)$ for each $W \in \hat{\Xi}_M$. But Λ is a $D\Delta(m_1, \ldots, m_t)$ lattice for some $t > 1$ and $m_i > 2$, so

$$\Lambda = \Lambda_1 * \cdots * \Lambda_t,$$

where $\Lambda_i \cong \Delta(m_i)$ and $(\Lambda'_i : 1 \leq i \leq t)$ are the connected components of Λ'. Therefore $\Xi_M = \Xi_1 * \cdots * \Xi_t$, where $\Xi_i = \Xi_M \cap \Lambda_i$, and with Ξ_i a sublattice of Λ_i containing $\mathcal{W}_0(\leq W)$ for each $W \in \Xi_i - \{\infty\}$. Hence the lemma follows from 1.3. □

4.6. *The following hold:*

(1) *For each* $V \in \hat{\Xi}_M$, $V\tilde{V} \in \hat{\Xi}_M$.

(2) *The map* $\varphi : V \mapsto V\tilde{V}$ *is a map of posets from* $\hat{\Xi}_M$ *into* $\hat{\Xi}_M$ *such that*

$$\varphi(V) \geq V.$$

(3) *If* Ξ_M *is disconnected or contains a connected component of* Λ', *then*

$$J = \varphi(J).$$

(4) *If* Ξ_M *contains a connected component* $\mathcal{C}$ *of* Λ', *then* $\varphi = 1$ *on* $\mathcal{C}$ *and*

$$\varphi(J) = J.$$

Proof. By 4.4.2, $\tilde{V}J \in \mathcal{W}_0$. Next $N_M(V)$ permutes the groups $\tilde{V}_E$, $E \in \mathcal{L}$, so $N_M(V)$ acts on $\tilde{V}$. If $H_* \leq \tilde{V}V$, then $H_{*D} \leq \tilde{V}V \cap D = \tilde{V}(V \cap D) = \tilde{V}$, contradicting $\tilde{V}J \in \mathcal{W}_0$. Thus $H_* \not\leq \tilde{V}V$, so (1) follows from 2.12.

Suppose $U \in \mathcal{W}_0$ with $U \leq V$. Then $U_D \leq V_D$, so for $E \in \mathcal{L}$, $\tilde{U}_E \leq \tilde{V}_E$ and hence $\tilde{U} \leq \tilde{V}$. Thus (2) holds.

If Ξ_M is disconnected or contains a connected component of Λ', then there exist $U_1, \ldots, U_r$ maximal in Ξ'_M such that $U_1 \wedge \cdots \wedge U_r = J$. By (2), $\varphi(U_i) = U_i$ and then

$$\varphi(J) = \varphi(U_1 \wedge \cdots \wedge U_r) = \varphi(U_1) \wedge \cdots \wedge \varphi(U_r) = U_1 \wedge \cdots \wedge U_r = J,$$

establishing (3).

Assume the hypothesis of (4). Then $\mathcal{C}$ is a connected component of Ξ_M and $\varphi : \Delta \to \Delta$, where $\Delta = \mathcal{C} \cup \{0\}$ by (2) and (3). Now (4) follows from (2) and the dual of 1.1 in [6] applied to $\varphi : \Delta \to \Delta$. □

5 The case G symmetric and H primitive

Hypothesis 5.1. Hypothesis 3.1 holds and G is the alternating or symmetric group on a set Ω.

In this section we assume:

Hypothesis 5.2. Hypothesis 5.1 holds and H is primitive on Ω. Set $S = \mathrm{Sym}(\Omega)$ and $A = \mathrm{Alt}(\Omega)$.

We will appeal to the theory of primitive subgroups of S developed in [4] and [5]. In particular from 2.2 and 2.3 in [4], there are five types of primitive groups: *affine*, *semisimple*, *diagonal*, *doubled*, and *complemented* primitive groups. Further there are various special classes of semisimple groups: *almost simple* groups, *product indecomposable* groups, and *octal groups*; these classes are not mutually exclusive.

5.3. *The following hold:*

(1) *H is not almost simple.*

(2) $J \neq 1$.

(3) *For $W \in \mathcal{W}_0$, HW is not almost simple.*

Proof. By 2.6, (1) is equivalent to (2). Suppose H is almost simple, so that $J = 1$. Let $V \in \mathcal{W}'_0$. Then $M = VH$ is primitive on Ω by 2.4 in [4], and $H \cap V = J = 1$, so V is a normal complement to H in M. In particular, M is not almost simple, so by Proposition 2 in [4], $H \cong L_3(2)$, $|\Omega| = 8$, and M is one of the two maximal parabolics of $G \cong L_4(2)$ which contain H. Thus we get $|\Lambda| = 4$, contradicting Λ a $D\Delta(m_1, \dots, m_t)$-lattice. This completes the proof of (1) and (2). Then (2) and 2.14 imply (3). □

5.4. $|\Omega|$ *is not prime.*

Proof. If so by 2.2 in [4], H is almost simple or affine. The first case does not hold by 5.3 and the second is out as H is not solvable. □

The notion of an (m,k)-*product structure* on Ω is defined in Definition 1.5 in [4]. An (m,k)-product structure can be thought of as a decomposition of Ω as a set product of k sets of order m, with $m \geq 5$ and $k \geq 2$. The stabilizers of product structures are maximal semisimple primitive subgroups of S. There is a partial ordering of product structures defined in Section 5 of [5].

The notion of a *pseudo-semisimple* subgroup of S is defined in Definition 5.7 of [5]. A pseudo-semisimple group is a primitive group K which preserves a product structure and the set of K-invariant product structures has a largest member $\mathcal{F}^+(K)$ (also defined in 5.7 of [5]).

5.5. *The following hold:*

(1) *Either* H *is affine and imprimitive on* $F^*(H)$*, or* H *is pseudo-semisimple.*

(2) *Let* $V \in \mathcal{W}_0$*. Then either* H *is affine, or* HV *is pseudo-semisimple.*

Proof. We first prove (1). By Hypothesis 5.2, H is primitive on Ω. From Definition 5.7 in [5], the primitive group H is pseudo-semisimple unless one of the following holds:

(i) H is almost simple.

(ii) H is affine.

(iii) H is doubled with two components.

(iv) H is strongly diagonal.

By 5.3, (i) does not hold. In case (ii) we may assume H is primitive on $F^*(H)$. In this case, and in cases (iii) and (iv), we show

$$\mathcal{M}(H) = \{N_G(F^*(H))\}, \qquad (*)$$

which contradicts 3.6.2.

If H is affine and primitive on $F^*(X)$, then $(*)$ follows from 5.4 and Proposition 4 in [4]. Similarly $(*)$ holds in cases (iii) and (iv) by Propositions 9 and 7 of [4], respectively. This completes the proof of (1).

Finally (2) follows from (1) and 5.8 in [5]. □

5.6. *For each $V \in \mathcal{W}_0$:*

(1) $F^*(HV) \leq V$.

(2) $F^*(HV)J \in \mathcal{W}_0$.

Proof. We first prove (1). By 5.3.2, $V \neq 1$, so V contains a minimal normal subgroup E of $M := HV$. By 2.4.1 in [4], M is primitive on Ω, so by 2.2 in [4], either $E = F^*(M)$ or M is doubled and $F^*(M) = E \times \tilde{E}$, with $\tilde{E} \cong E$, and we may assume the latter, with $V \cap \tilde{E} = 1$. Let $M^* := M/V$. As $V \in \mathcal{W}_0$, it follows that $F^*(M^*) = H_0^* \cong L$ is a nonabelian simple group. But as $\tilde{E}^* \cong \tilde{E}$ is the direct product of simple groups, and as $\tilde{E}^* \trianglelefteq H^* = M^*$, we have

$$\tilde{E}^* \trianglelefteq F^*(M^*) = H_0^*,$$

so $\tilde{E}$ is simple. This is a contradiction as M is pseudo-semisimple by 5.5.2, so $\tilde{E}$ has more than one component. This completes the proof of (1). Then (1) and 2.7.2 imply (2). □

See 5.7 in [5] for the definition of the product structure $\mathcal{F}^+(X)$ of a pseudo-semisimple subgroup X of S.

5.7. *If $V \in \mathcal{W}_0$ and HV is pseudo-semisimple, then $\mathcal{F}^+(HV)$ is the greatest member of $\boldsymbol{\mathcal{F}}(HV)$.*

Proof. This is 5.9.1 in [5]. □

Notation 5.8. If $V \in \mathcal{W}_0$ and HV is pseudo-semisimple, set

$$M(V) := M(\mathcal{F}^+(HV)),$$

where $M(\mathcal{F})$ is the stabilizer in G of the product structure $\mathcal{F}$. Set

$$X(V) := F^*(M(V)),$$

and write $K(V)$ for the kernel of the action of $M(V) \cap \hat{G}$ on the components of $M(V)$. This makes sense by 5.7.

5.9. *Assume $V \in \mathcal{W}_0$ and HV is pseudo-semisimple but not affine. Set*

$$D := F^*(HV).$$

Then:

(1) $M(V) \in \mathcal{M}(HV)$.

(2) $F^*(HV) \leq X(V)$.

(3) *Assume HV is semisimple and product indecomposable. Then each component D_1 of HV is contained in a component X_1 of $X(V)$. Moreover*

$$D_1 = F^*(HV \cap X_1) \quad \text{and} \quad C_{X_1}(D_1) = 1.$$

(4) *Assume HV is semisimple and product indecomposable. Then $N_{X(V)}(D)/D$ is solvable and $N_{X(V)}(D)J$, $X(V)J$, and $K(V)J$ are in $\mathcal{W}_0$.*

Proof. Part (1) follows from the definition of $M(V)$ in 5.8 and Proposition 5 in [4] which says that $M(V) \in \mathcal{M}$. Part (2) follows from 4.3 in [4]; for example in case (2) of 4.3 in [4], the overgroup M is not the stabilizer $M(V)$ of a product structure.

Assume HV is semisimple and product indecomposable. The first remark in (3) follows from Proposition 5 in [4]; the structure of $HV \cap X_1$ and the fact that $C_{X_1}(D_1) = 1$ are consequences of the fact that $\mathcal{F}^+(M(V)) = \mathcal{F}^+(HV)$.

Let $X_1, \dots, X_r$ be the components of $X := X(V)$. Set $F_i := F^*(HV \cap X_i)$. We must show that $N_X(D)/D$ is solvable. But D is the direct product of the subgroups F_i and $N_X(D)$ is the direct product of the subgroups $N_{X_i}(F_i)$, so it remains to show that $N_{X_1}(F_1)/F_1$ is solvable. But (3) says that $F_i = D_i$ is simple, so $\mathrm{Out}_{X_i}(F_i)$ is solvable by the Schreier property. Hence $N_X(D)/D$ is solvable, and then XJ and $N_X(D)J$ are in $\mathcal{W}_0$ by 2.9. Then we have $K(V)J \in \mathcal{W}_0$ by 2.8 as $K(V)/X$ is solvable. This completes the proof of (4). □

For H affine with $F^*(H) = D$, recall the definition of $\mathcal{D}(H)$ and $\mathcal{F}(\mathcal{D})$ for $\mathcal{D} \in \mathcal{D}(H)$, from 2.6 and 1.6 in [4].

5.10. *Assume H is affine, set $D := F^*(H)$, and for $\mathcal{D} \in \mathcal{D}(H)$ let*

$$M(\mathcal{D}) := M(\mathcal{F}(\mathcal{D})) \quad \text{and} \quad X(\mathcal{D}) := F^*(M(\mathcal{D})).$$

Set $\mathcal{W}_1^* := \{V \in \mathcal{W}_0^* : V \not\leq N_G(D)\}$. Then:

(1) $\mathcal{M}(H) = \{N_G(D), M(\mathcal{D}) : \mathcal{D} \in \mathcal{D}(H)\}$.

(2) $\mathcal{W}_1^* \neq \varnothing$.

(3) *For $V \in \mathcal{W}_1^*$, HV is semisimple, product indecomposable, and pseudo-semisimple, and*

$$F^*(HV) = X(V),$$

where $X(V) := X(\mathcal{E}(V))$ for some $\mathcal{E}(V) \in \mathcal{D}(H)$.

(4) $X(V)J = V$.

(5) $N_{X(V)}(D) \leq J$.

(6) $D = F^*(J)$.

Proof. Part (1) follows from 5.4 and Proposition 4 in [4], while part (2) follows from 3.3.

Let $V \in \mathcal{W}_1^*$ and set $Q := F^*(HV)$. Then $HV \in \mathcal{O}_G(H)'$, so by 5.4 and Proposition 4 in [4], $D \leq Q$ and either $Q = D$ or HV is semisimple and

$$\mathcal{F}(HV) = \mathcal{F}(\mathcal{D})$$

for some $\mathcal{D} \in \mathcal{D}(H)$. As $V \not\leq N_G(D)$, the latter holds. Set $\mathcal{E}(V) := \mathcal{D}$.

By 5.6, $\bar{Q} \in \mathcal{W}_0$ and $\bar{Q} \leq V$. As $D \neq Q$, $Q \not\leq J$, so $V = \bar{Q}$ as $V \in \mathcal{W}_0^*$.

As H is affine, $|\Omega|$ is a prime power, so HV is product indecomposable (cf. Definition 5.10 in [4]). Hence (cf. Definition 5.7 in [5]) HV is pseudo-semisimple and $\mathcal{F}^+(HV) = \mathcal{F}(HV) = \mathcal{F}(\mathcal{D})$, so in the notation of 5.8,

$$M := M(\mathcal{D}) = M(V) \quad \text{and} \quad X(\mathcal{D}) = X(V).$$

Set $X := X(V)$ and $K := K(V)$. As $X = X(\mathcal{D})$, it follows from 1.6 in [4] that $X = X_1 \times \cdots \times X_r$ is the direct product of its components and $D = D_1 \times \cdots \times D_r$, where $D_i = D \cap X_i$. By 5.9.4, $\bar{X}$ and $\bar{K}$ are in $\mathcal{W}_0$, and by 5.9.2, $Q \leq X$, so $V = \bar{Q} \leq \bar{X}$. As $\mathcal{F}(HV) = \mathcal{D}$ and HV is semisimple and product indecomposable, 5.9.3 says that $Q = Q_1 \times \cdots \times Q_r$, where $Q_i = X_i \cap Q$ is a component of Q.

Suppose $X \neq Q$. Then $V < \bar{X} \leq \bar{K}$, so applying 3.7 to X, KH in the roles of Y, X it follows that $KH = XN_{KH}(Q)$. Also $D_1 \leq Q_1 < X_1$, so by 3.2.2 in [4], $|D_1| = p$ is prime. As $KH = XN_{KH}(Q)$, we get $K = X_1 N_K(Q_1)$. But now as $\mathrm{Aut}_K(X_1) \cong S_p$, 5.7 in [14] contradicts $Q_1 < X_1$. Therefore $X = Q$, completing the proof of (3). Further $V = \bar{Q} = \bar{X}$, establishing (4).

Next for $Y \in \mathcal{I}_{N_X(D)}(H)$, $\bar{Y} \in \mathcal{W}_0$ by 2.7.1. Then as $V = \bar{X} \in \mathcal{W}_0^*$, part (5) follows.

By 5.6, $D \leq J$. As $D = F^*(H)$ and $D \leq J \trianglelefteq H$, it follows that $D = F^*(J)$, establishing (6). □

5.11. *Assume H is affine. Then:*

(1) *There exists a unique $V \in \mathcal{W}_1^*$.*

(2) *Choose $\mathcal{E}(V)$ as in* 5.10.3. *Then $\mathcal{E}(V)$ is the unique member of $\mathcal{D}^*(H)$.*

Proof. Let $D := F^*(H)$ and $M := N_G(D)$. By 5.10.2 there exists a $V \in \mathcal{W}_1^*$. Let $\mathcal{E} := \mathcal{E}(V)$, $X := X(V)$, $K := K(V)$, and $P := N_X(D)$. By 5.10.5, $P \leq J$. But from 4.1 in [4], $N_K(D)$ is the kernel of the action of $N_M(\mathcal{E})$ on $\mathcal{E}$. Thus as $N_K(D)/P$ is a 2-group of exponent 2, it follows from 4.7 in [5] that either $\mathcal{E}$ is the unique maximal P-invariant member of $\mathcal{D}(H)$, and hence (2) holds, or $\mathcal{D}$ consists of k members of order 5, and choosing $\mathcal{E}' \in \mathcal{D}(H)$ with $\mathcal{E}' \not\leq \mathcal{E}$, $\mathcal{E}'$ also has k members of order 5 and $\mathcal{B} := \mathcal{E} \wedge \mathcal{E}' = \{B_1, \ldots, B_{k/2}\}$ has $k/2$ members of order 25.

Assume the latter case and let $X' := X(\mathcal{E}')$. Then $P' := N_{X'}(D)$ is solvable, so by 2.8, $V' := \bar{X}' \in \mathcal{W}_0$. Further arguing as in the proof of 5.10.5, $P' \leq J$, so as P' is a maximal H-invariant subgroup of X', we conclude $V' \in \mathcal{W}_0^*$.

Suppose that $k = 2$. Then

$$N_{F^*(G)}(D) \cong (\boldsymbol{Z}_4 * \mathrm{SL}_2(5))/E_{25}$$

and $H \leq N_G(D)$ acts on J and X with $N_X(D) \cong D_{10} \times D_{10}$ contained in J, so $N_G(J)/J$ is solvable, a contradiction.

Hence $k > 2$, so that $\mathcal{B} \in \mathcal{D}(H)$. Moreover there is a third member $\mathcal{E}''$ of $\mathcal{D}(H)$ with $\mathcal{B} = \mathcal{E} \wedge \mathcal{E}'' = \mathcal{E}' \wedge \mathcal{E}''$, and by symmetry $V'' = \bar{X}'' \in \mathcal{W}_0$, where $X'' := X(\mathcal{E}'')$. Let $Y := X(\mathcal{B})$ and Y_1 a component of Y. From 5.12 in [5], we have $U \leq Y$ for $U \in \{X, X', X''\}$, and $N_{Y_1}(U)$ is the stabilizer in Y_1 of a regular product structure on ωY_1. In particular Y_1 contains components $U_j \cong A_5$ of U, $j \in \{1, 2\}$, and $Y_1 \cong A_{25}$ with $N_{Y_1}(U)/U_1U_2$ solvable. Applying 2.8 to $X \cap Y$, $N_Y(X)$, Y in the roles of X_1, X_2, X_3, we conclude that $\bar{Y} \in \mathcal{W}_0$. Then by 2.7, $N_Y(U)J \in \mathcal{W}_0$ for $U \in \{X', X''\}$. As $N_{Y_1}(U)$ is the stabilizer of a regular product structure, it is maximal in Y_1, so $Y = \langle X, N_Y(U)\rangle$. As $V = \bar{X} \in \mathcal{W}_0^*$ and $N_Y(X') \neq N_Y(X'')$, this contradicts 3.4 and 3.7. Therefore (2) holds.

If $V' \in \mathcal{W}_1^*$, then by (2) applied to V', $\mathcal{E}(V') = \mathcal{E}$, so by 5.10.4, $V' = \bar{X} = V$, establishing (1). □

5.12. *H is not affine.*

Proof. Assume H is affine and set $D := F^*(H)$, $M := N_G(D)$, $M^* := M/D$. By 5.11.1 there is a unique $V \in \mathcal{W}_1^*$. Let $\mathcal{E} := \mathcal{E}(V)$. As Λ is disconnected, there exists $W \in \mathcal{W}_0^*$ with $\mathcal{C}(W) \neq \mathcal{C}(V)$. Then $W \notin \mathcal{W}_1^*$, so $W \leq M$. Set $P := HW$.

Let $Q := J \cap X(V)$ and Q' the kernel of the action of $N_M(\mathcal{E})$ on $\mathcal{E}$. By 4.1 in [4], $Q' = N_{K(V)}(D)$. By 5.10.5, $Q \geq N_{X(V)}(D)$, so Q'/Q is a 2-group of exponent 2.

Suppose that P is primitive on D. As $H^* \leq P^*$ with H^* not solvable, and as Q'/Q is a 2-group of exponent 2, 4.8 and 4.9 in [5] say that $F^*(P^*) = P_0^* Z^*$ where P_0^* is quasisimple and Z^* induces scalars on D. Thus as

$$1 \neq Q^* \leq W^* \trianglelefteq P^*,$$

$P_0^* \leq W^*$, so P^*/W^* is solvable, contradicting $F^*(H/W) \cong L$.

Therefore there exists a $\mathcal{D} = \{D_1, \ldots, D_m\} \in \mathcal{D}(P)$. Suppose $\mathcal{D} = \mathcal{E}$. Then $P \leq M(V)$ and as $\mathcal{C}(V) \neq \mathcal{C}(W)$, 3.8 and 3.9.3.d in [6] say that $V = \theta(W)$ acts on J, contrary to $V \in \mathcal{W}_1^*$ and 5.10.6. Thus $\mathcal{D} \neq \mathcal{E}$, so by 5.11.2, $\mathcal{D} < \mathcal{E}$. Let R be the kernel of the action of $N_M(\mathcal{D})$ on $\mathcal{D}$, $R_P := R \cap P$, and $Y := X(\mathcal{D})$. From 4.1 in [4], $H \cap Y \leq P \cap Y \leq N_Y(D) \leq R$.

As $W \in \mathcal{W}_0^*$, it follows that H is maximal in P, so $\mathcal{D}$ is maximal in $\mathcal{E}$. Therefore as Q'/Q is a 2-group of exponent 2, 4.10 in [5] says that either $H \cap R$ is solvable or $F^*(R_P^*) = R_0^* Z^*$ where Z^* induces scalars on each D_i, $R_0^* = R_1^* \cdots R_m^*$ with H transitive on the R_i^*, and for $1 \leq i \leq m$, R_i^* is quasisimple and R_i is contained in the ith component of Y. As $H \cap Y \leq H \cap R$, in the first case $\bar{Y} \in \mathcal{W}_0$ by 2.8. Similarly in the second case, as $1 \neq Q^* \leq R_0^*$, we get $R_0 = \langle Q^P \rangle \leq W$, so $R/(R \cap W)$ is solvable. Applying 2.8 to $Y \cap W$, $Y \cap P$, Y in the role of X_1, X_2, X_3, we again conclude that $\bar{Y} \in \mathcal{W}_0$. As $\mathcal{D} < \mathcal{E}$, $\mathcal{F}(\mathcal{D}) < \mathcal{F}(\mathcal{E})$, so $X(V) < Y$ by 5.12.1 in [5]. But now $\mathcal{C}(V) = \mathcal{C}(\bar{Y})$, and arguing as in the previous paragraph, $\mathcal{C}(\bar{Y}) = \mathcal{C}(W)$, contrary to the choice of W. □

5.13. *If H is semisimple, then H is not octal.*

Proof. See 4.2 in [4] for the definition of an octal semisimple group. Assume H is octal, and set $X := X(J)$, $M := M(J)$, and $K := K(J)$. By 5.9.4, $\bar{X}$ and $\bar{K}$ are in $\mathcal{W}_0$. Let D_1 be a component of H. As H is octal, from 4.2 in [4], it follows that $D_1 \cong \mathrm{Aut}_H(D_1) \cong L_3(2)$ and $\Gamma_1 := \omega D_1$ is of order 8. By 5.9.3, D_1 is contained in a component X_1 of X. As $|\Gamma_1| = 8$, $X_1 \cong A_8 \cong L_4(2)$. Now (cf. Proposition 2 in [4]) $X \cap J = F^*(H) = D$ and $\mathcal{V}_X(H) = \{D, Y_1, Y_2, X\}$, where Y_i is affine with $Y_{i,1} = Y_i \cap X_1 \cong L_3(2)/E_8$ a maximal parabolic of X_1. By 2.7, we have $\bar{Y}_i \in \mathcal{W}_0$. As $Y_1 \cap Y_2 = D = X \cap J$, it follows from 3.4 that $\bar{Y}_i \in \mathcal{W}_0^*$ and $\bar{X} = \bar{Y}_1 \vee \bar{Y}_2$. But $X < K$, and as $\mathrm{Aut}_H(D_1) = D_1$, $N_K(D) = D$. As K/X is solvable, $\bar{K} \in \mathcal{W}_0$ by 2.8. As $N_K(D) = D = X \cap J$, 3.7.4 supplies a contradiction. □

5.14. *We have $\mathcal{M}(H) = \{M(\mathcal{F}) : \mathcal{F} \in \boldsymbol{\mathcal{F}}(H)\}$, and $\mathcal{F}^+(H)$ is the greatest member of $\boldsymbol{\mathcal{F}}(H)$.*

Proof. By 5.12, H is not affine, so by 5.5.2, H is pseudo-semisimple. By 5.13, H is not octal semisimple. Thus the lemma follows from 5.17 in [5] and 5.7. □

5.15. *The following hold:*

(1) *H is not doubled, complemented, or diagonal.*

(2) *If H is semisimple, then H is product indecomposable.*

Proof. See 2.3 in [4] for the definition of doubled, complemented, and diagonal primitive groups. See 5.10 in [4] for the definition of product decomposable primitive groups. Assume H is a counterexample to (1) or (2), and set $D := F^*(H)$, $X := X(J)$, $M := M(J)$, and $\mathcal{F} := \mathcal{F}^+(H)$. By 3.3 there exists $V \in \mathcal{W}_0^*$ with $V \not\leq M$. By 5.12 and 5.5.2, HV is pseudo-semisimple, so $\mathcal{F}^+(HV)$ exists by 5.7.

Then by 5.14, $\mathcal{F}^+(HV) < \mathcal{F}$, so by 5.14 in [5] we get $X \le X(HV) =: Y$. As $V \not\le M = N_G(X)$, $X \ne Y$, and hence $\mathcal{F} \ne \mathcal{F}^+(HV)$.

Claim HV is semisimple. If H is semisimple, then this is a consequence of Proposition 5 in [4], and the fact that H is not octal by 5.13. So we may assume H is doubled, complemented, or diagonal. If $D = F^*(HV)$, or H is complemented and $F^*(HV) = DC_G(D)$, then from Definition 5.7 in [5] and Notation 2.6 in [4], $\mathcal{F} = \mathcal{F}^+(H) = \mathcal{F}(H) = \mathcal{F}(HV) = \mathcal{F}^+(HV)$, contrary to the remark at the end of paragraph one of the proof. Thus by Propositions 7, 9, and 11 in [4], HV is semisimple. This completes the proof of the claim.

We next claim that $F^*(HV) = Y$. Assume otherwise. By 5.8.3 in [4], the parameter b of Notation 5.2 in [4] is 1 or 2. Suppose $b = 2$. Then by 5.9.3 and by Definition 5.10 and 5.8.4 in [4], H is product decomposable, while by 5.11 in [4], $\mathcal{F}^+(HV) = \mathcal{F}$, contrary to an earlier remark. Thus $b = 1$. As $H < HV$, we conclude from 5.5.5 in [4] that H is semisimple and the parameter s of Notation 5.2 in [4] is equal to 1. As $b = 1$, a component D_1 of D is contained in a component L_1 of HV, and then as $s = 1$, D_1 is transitive on $\Gamma_1 := \omega L_1$. As H is semisimple and a counterexample to (1) or (2), H is product decomposable. Now by the Main Theorem of [12], either $N_{\mathrm{Sym}(\Gamma_1)}(D_1)$ is the unique maximal overgroup in $\mathrm{Sym}(\Gamma_1)$ of D_1, so that $L_1 = D_1$, or $D_1 \cong \mathrm{Sp}_4(q)$ and $L_1 \cong \mathrm{Sp}_{4e}(q^{1/e})$ for some e. In the first case $D = F^*(HV)$ so HV is also product decomposable and $\mathcal{F}^+(HV) = \mathcal{F}$, a contradiction. The second case is impossible as some element of H induces an automorphism on D_1 nontrivial on the Dynkin diagram of D_1, and such automorphisms do not lift to L_1. This completes the proof of the second claim.

By the second claim and 5.6.2, $\bar{Y} \in \mathcal{W}_0$. Then $\bar{X} \in \mathcal{W}_0$ by 2.7. Now as $X < Y$, Y does not act on X. As M is the stabilizer of a product structure, M is semisimple and product indecomposable. Then as H is not, $D < X$, so X does not act on D. Then as J does act on D and X, it follows that $J = \bar{D} \ne \bar{X} \ne \bar{Y}$. Then $V \in \mathcal{W}_0^*$ and $J < \bar{X} < \bar{Y} = V$, so we have a contradiction. □

5.16. *H is semisimple, product indecomposable, and not octal.*

Proof. By 5.12, H is not affine, so by 5.15.1, H is semisimple. Then by 5.13, H is not octal, and by 5.15.2, H is product indecomposable, completing the proof. □

Notation 5.17. Adopt Notation 5.8, and let $\mathcal{F} := \mathcal{F}^+(H)$ and $X := X(J)$. Let $\Xi := \boldsymbol{\mathcal{F}}(H) - \{\mathcal{F}\}$ and Ξ^* the maximal members of Ξ. Set

$$\mathcal{W}_1^* := \{V \in \mathcal{W}_0^* : \mathcal{F}^+(HV) \in \Xi^*\}.$$

Let $I := \{1, \dots, r\}$ and $\mathcal{X} := \{X_i : i \in I\}$ be the set of components of X. Set $m := |\omega X_1|$ for $\omega \in \Omega$. Thus $\mathcal{F}$ is a regular (m, r)-product structure on Ω.

Represent $N_S(\mathcal{F})$ on I so that the map $\varphi : i \mapsto X_i$ is an equivalence of that representation with the representation of $N_S(\mathcal{F})$ on $\mathcal{X}$ via conjugation. For given $Y \leq N_S(\mathcal{F})$, write Y^I for the image of Y in $\mathrm{Sym}(I)$ under this representation.

5.18. *The following hold:*

(1) $X = F^*(H) = F^*(J)$.

(2) $K(J) \leq J$.

(3) *For $V \in \mathcal{W}_0$, HV is semisimple, product indecomposable, and not octal; moreover $X \leq X(V)$, and $K(J) \leq K(V)$.*

(4) $(M(J) \cap A)^I = \mathrm{Sym}(I)$.

Proof. Let $V \in \mathcal{W}_0$. By 5.16 and Proposition 5 in [4], HV is semisimple and product indecomposable but not octal. This allows us to verify the hypothesis of 2.10 with $M(V)$ and $K(V)$ defined in 5.8. Then we appeal to that lemma.

Hypothesis (a) of 2.10 follows from 5.6.1. As $F^*(M(V))$ is perfect, we have $F^*(M(V)) \leq F^*(G)$. By 5.9.1, $M(V) \in \mathcal{M}(HV)$, while by 5.9.2, $F^*(H(V)) \leq X(V)$. Then by 5.9.4, the remaining condition in hypothesis (b) of 2.10 is satisfied.

As H is semisimple, $F^*(H)$ is the unique member of the set $\mathcal{X}$ of 2.10.2. As $M(J) = M(\mathcal{F}^+(H))$, we get $N_G(F^*(H)) \leq M(J)$. By 5.14, and by 5.14 in [5], $X \leq X(V)$, so hypothesis (c) of 2.10 is satisfied, and the second statement in (3) holds. Hence as Λ is disconnected, $X \leq J$ by 2.10.2. Then as $J \trianglelefteq H \leq M(J)$ and $X = E(X) = F^*(M(J))$, (1) holds.

By construction, $X(V) \leq K(V) \trianglelefteq M(V)$ with $K(V) \leq \hat{G}$, and $K(V)/X(V)$ is a 2-group. As $\mathcal{F} \leq \mathcal{F}(HV)$, $K(J) \leq K(V)$, completing the proof of (3). Now (2) follows from 2.10.3.

By (2), $H \cap K(J) \leq J$, so as H/J is not solvable, $r > 2$. Thus (4) follows from 5.10.3 in [5]. $\square$

5.19. *Let $V \in \mathcal{W}_0$. Then:*

(1) *HV is semisimple, product indecomposable, and not octal.*

(2) $X \leq X(V) = F^*(HV)$.

(3) *If $V \in \mathcal{W}_0^*$, then either*

  (i) *$X(V) = X$ and H is maximal in HV, or*

  (ii) *$V = X(V)J$, $V \in \mathcal{W}_1^*$, and $N_{X(V)}(X) \leq J$.*

(4) $K(V)J \in \mathcal{W}_0$.

Proof. By 5.18.1, $X = F^*(H)$. For $P \le G$ let $m(P) := |\omega P|$. A component X_1 of X is isomorphic to the alternating group of degree $m := m(X_1)$.

We first prove (1) and (2). By 5.10.3, HV is semisimple and product indecomposable, but not octal, and by Proposition 5 in [4], X_1 is contained in a component Y_1 of $Y := F^*(HV)$. By 5.9.3, Y_1 is contained in a component D_1 of $X(V)$ and $Y_1 = F^*(HV) \cap D_1$, so $m(Y_1) = m(D_1)$. Let s be the number of components of H contained in Y_1; this is the parameter defined in Notation 5.2 of [4]. By 5.5.5 in [4], either $Y_1 \cong A_{m(Y_1)}$, or $s = 1$. In the former case as $m(Y_1) = m(D_1)$, Y_1 is a component of $X(V)$ and statement (1) holds, so assume $s = 1$. Then $m(Y_1) = m$, and $X_1 \le Y_1 \le D_1$, with $D_1 \cong A_{m(Y_1)}$, so as $X_1 \cong A_m$, $X_1 = Y_1 = D_1$. This completes the proof of (1) and (2).

We next prove (3), so assume $V \in \mathcal{W}_0^*$.

By 2.12.1 in [6], H is maximal in HV. Thus (3i) holds if $X = X(V)$, so we may assume $X(V) \ne X$. By (2) and 5.6, $X(V) \le V$ and $V' := \overline{X(V)} \in \mathcal{W}_0$, so that $V' \le V$. Further if $\mathcal{F}^+(HV) \le \mathcal{D} < \mathcal{F}$, then by 5.14 in [5], $X(\mathcal{D}) \le X(V)$, so $V'' := \overline{X(\mathcal{D})} \in \mathcal{W}_0$ by 2.7. Thus as $V \in \mathcal{W}_0^*$, $V' = V = V''$, so $V = \overline{X(V)}$ and $\mathcal{F}^+(HV) \in \Xi^*$. Hence $V \in \mathcal{W}_1^*$. Further by 2.7, $N_{X(V)}(X)J \in \mathcal{W}_0$, so as $V \in \mathcal{W}_0^*$, $N_{X(V)}(X) \le J$. This shows that if $V \in \mathcal{W}_0^*$ and (3i) fails, then (3ii) holds, completing the proof of (3).

As $\overline{X(V)} \in \mathcal{W}_0$ and $K(V)/X(V)$ is solvable, (4) follows from 2.8. □

5.20. $\mathcal{W}_1^* \ne \varnothing$.

Proof. Assume otherwise. Then by 5.19.3, $\mathcal{W}_0^* \subseteq M(J)$, contrary to 3.3. □

5.21. *Let $V \in \mathcal{W}_1^*$. Let $\mathcal{X}(V)$ be the set of components of $X(V)$, and for $Y \in \mathcal{X}(V)$ set $\sigma(Y) = \{i \in I : X_i \le Y\}$ and $k = k(V) := |\sigma(Y)|$. Set $P(V) := N_{K(V)}(X)$. Then:*

(1) $\Sigma = \Sigma(V) := \{\sigma(Y) : Y \in \mathcal{X}(V)\}$ *is an H-invariant partition of I such that* $N_{M(J)}(\Sigma) = M(J) \cap M(V)$.

(2) $P(V)^I$ *is the kernel $K(\Sigma)$ of the action of $N_{\mathrm{Sym}(I)}(\Sigma)$ on Σ.*

(3) *Either $N_{X(V)}(X)^I = K(\Sigma)$ or $k = 2$, $m \equiv 2 \mod 4$, $N_{X(V)}(X) = K(J)$, and the involution $\tau = \tau(V) \in \mathrm{Sym}(I)$ with cycles (i, j) for $\{i, j\} \in \Sigma$ is in J^I.*

(4) *Let $U \in \mathcal{W}_0$ and set $Q := N_{HU}(X)^I$. Assume either*

   (a) $P(V) \le U$, *or*

   (b) $N_{X(V)}(X)^I = K(\Sigma)$.

   Then Q is imprimitive on I, and for each nontrivial Q-invariant partition Γ of I, $\Gamma \le \Sigma$.

(5) $\mathcal{W}_1^* = \{V\}$.

Proof. Set $T := N_S(X(V)) \cap N_S(X)$. Visibly Σ is a T-invariant partition of I. By 5.14, $\mathcal{F}^+(HV) \leq \mathcal{F}$, so by 5.12.2 in [5], T is the stabilizer in $N_S(X)$ of a nontrivial partition Σ' of I. By 1.8 in [4], $N_S(\mathcal{F})^I = \mathrm{Sym}(I)$, so Σ' is the unique nontrivial T-invariant partition of I, and hence $\Sigma = \Sigma'$ and

$$N_{M(J)}(\Sigma) = M(J) \cap M(V),$$

establishing (1).

Next $P(V)$ acts on each block of Σ so $P(V)^I \leq K(\Sigma)$. On the other hand by (1),

$$N_{M(J)}(\Sigma) = M(J) \cap M(V),$$

so $M(J)_\Sigma$ is the kernel of the action of $M(J) \cap M(V)$ on Σ. Then (2) follows from 5.18.4.

By 5.12.4 in [5], T is the subgroup of $N_S(X(V))$ permuting the set $\mathcal{C}$ of product structures defined in that lemma. Hence for $Y \in \mathcal{X}(V)$, $N_Y(X)$ is the stabilizer of the (m,k)-product structure on the m^k-set permuted by Y. Then all but the last remark in part (3) follow from 5.10.3 in [5], which says the $Y^\alpha = \mathrm{Sym}(Y)$, where $\alpha := \{X_i : X_i \leq Y\}$, unless the exceptional case in (3) holds. Further in that event, there is $t \in K(V)$ with $t^I = \tau$ the involution in (3). Set $B := \langle t \rangle X(V)$. Then B is H-invariant, so by 5.19.4 and 2.7, $\bar{B} \in \mathcal{W}_0$. Further if $B \neq (B \cap J)X(V)$, then by 3.7, $|\bar{B} : \overline{X(V)}| = 2$, contrary to 3.5.2. Thus we may take $t \in J$, completing the proof of (3).

Assume the hypothesis and setup of (4). As $V \in \mathcal{W}_1^*$, $N_{X(V)}(X) \leq J$ by 5.19.3. In case (4b), $K(\Sigma) = N_{X(V)}(X)^I \leq J^I \leq U^I$, while in case (4a), $P(V) \leq U$ by hypothesis, and by (2), $K(\Sigma) = P(V)^I$. Therefore in either case,

$$K(\Sigma) \leq U^I \leq Q.$$

Thus if Q is primitive on I, then by 3.2.1 in [8], Q is the symmetric group on I and $Q = U^I$. This is impossible as the kernel $K(J)$ of the action of $M(J)$ on I is contained in J by 5.18.2, and $H \leq N_{HU}(X)$ with $H \cap U = J$ and $F^*(H/J) \cong L$. Therefore Q is imprimitive on I.

Pick Γ as in (4). As $K(\Sigma) \leq Q^I$, 3.2.2 in [8] shows that $\Gamma \leq \Sigma$, completing the proof of (4).

Suppose $V' \in \mathcal{W}_1^* - \{V\}$. Then by (3) and (4) applied to V', $\Sigma' = \Sigma(V')$ in the role of U, Γ, either $k = 2$ or $\Sigma' \leq \Sigma$. Assume $k > 2$. If $k(V') = 2$ then Σ' is a maximal partition of I, so $\Sigma' = \Sigma$. On the other hand if $k(V') > 2$, then by symmetry between V and V', $\Sigma \leq \Sigma'$, so again $\Sigma = \Sigma'$. Thus in any event, $\mathcal{F}^+(HV) = \mathcal{F}^+(HV')$ by (1) and 5.12.2 in [5], and then $V = X(\mathcal{F}^+(HV)) = X(\mathcal{F}^+(HV')) = V'$, a contradiction.

So we may assume $k = 2$, and then by symmetry also $k(V') = 2$. Now we have $[\tau, \tau(V')] = 1$ and H^I centralizes $E^I = \langle \tau, \tau(V') \rangle$, so H preserves the partition Φ of I consisting of the orbits of E^I on I. Moreover the kernel $K(\Phi)$ of the action of $M(J)^I = \mathrm{Sym}(I)$ on Φ is a direct product $K(\Phi) = K_1 \times \cdots \times K_s$, where $K_i \cong S_4$, $K_i = \langle K_i \cap P(V)^I, K_i \cap P(V')^I \rangle$, $K_i \cap P(V)^I \cong K_i \cap P(V')^I \cong E_4$. Let R be the preimage of $K(\Phi)$ in $M(J)$. Then $R/K(J)$ is solvable, so $\bar{R} \in \mathcal{W}_0$ by 2.8. Also $\bar{P}(V)$ and $\bar{P}(V') \in \mathcal{W}_0$. Finally $N_H(K_1)$ acts on the third 4-subgroup $\bar{P}_1^I$ of K_1 generated by transpositions, and hence on its preimage P_1 in $M(J)$, and on $P = \langle P_1^H \rangle$. Thus $\bar{P} \in \mathcal{W}_0$ by 2.7. Next

$$P(V)^I \cap P(V')^I = O_2(K(\Phi)) = N_{K(\Phi)}(\Sigma) \cap N_{K(\Phi)}(\Sigma'),$$

so $J \cap R$ is contained in the preimage R_0 of $O_2(K(\Phi))$ in R. As

$$P(V) \cap P(V') = P(V) \cap P = P(V') \cap P = R_0,$$

this is contrary to 3.7. Thus (5) holds. □

5.22. *Let $V \in \mathcal{W}_1^*$ and set $P(V) := N_{K(V)}(X)$. Assume $P(V) \le U \in \mathcal{W}_0$ with $U \le M(J)$, and set $Q := (HU)^I$. Then there exists $\mathcal{D} \in \boldsymbol{\mathcal{F}}(HU)$ such that*

$$\mathcal{D} \le \mathcal{F}^+(HV) \quad \textit{and} \quad X(V) \le X(\mathcal{D})$$

Proof. By hypothesis,

$$P(V) \le U \le M(J),$$

so we can form U^I, and $P(V)^I \le U^I$. Define $\Sigma = \Sigma(V)$ as in 5.21. By 5.21.4, there is a nontrivial Q-invariant partition Γ of I and $\Gamma \le \Sigma$. As Γ is HU-invariant, it determines $\mathcal{D} := \mathcal{F}(H, \Gamma) \in \boldsymbol{\mathcal{F}}(HU)$ by 2.6 in [4]. As $\Gamma \le \Sigma$, it follows that $\mathcal{D} \le \mathcal{F}^+(HV)$, and hence $X(V) \le X(\mathcal{D})$ by 5.14 in [5]. This completes the proof of the lemma. □

Theorem 5.23. *Assume Hypothesis* 5.1. *Then H is not primitive on* Ω.

Proof. Assume otherwise. Then Hypothesis 5.2 is satisfied, so we can appeal to the results in this section. In particular, appealing to 5.20 and 5.21.5, there is a unique $V \in \mathcal{W}_1^*$. Let $\mathcal{C} = \mathcal{C}(V)$ be the connected component of Λ containing V and adopt the notation of 3.4. Set $\alpha := \mathcal{C}^* - \{V\}$ and $U := L_\alpha$.

As $\mathcal{W}_1^* = \{V\}$, each $W \in \mathcal{W}_0^* - \{V\}$ satisfies 5.19.3.i. In particular, we have $X(W) = X$, so by 5.19.2, $X = F^*(HW)$, so $W \le M(J)$. Thus $U \le M(J)$.

On the other hand as $V \in \mathcal{W}_1^*$, it follows that V satisfies 5.19.3.ii, so $V = \overline{X(V)}$ and $N_{X(V)}(X) \le J$. As in 5.22, set $P(V) := N_{K(V)}(X)$. Then

$$P(V) \cap X(V) = N_{X(V)}(X) \le J,$$

so $P(V) \le U$ by 3.4. Thus the hypotheses of 5.22 are satisfied, so we may choose $\mathcal{D}$ and $D := X(\mathcal{D})$ as in that lemma. Thus $\mathcal{D} \in \boldsymbol{\mathcal{F}}(HU)$ so $U \le M(\mathcal{D})$, while $X(V) \le D$ by 5.22 so $V = \overline{X(V)} \le M(\mathcal{D})$. Therefore $\mathcal{C}^*$ acts on D, contrary to 3.8. This completes the proof of the theorem. □

6 The transitive but imprimitive case

In this section we assume:

Hypothesis 6.1. Hypothesis 5.1 holds with H transitive but not primitive on Ω. Set

$$\mathcal{P}_1 = \mathcal{P}_1(H) := \{\Gamma \in \mathcal{P}'(H) : H_* \le G_\Gamma\},$$
$$\mathcal{P}_2 = \mathcal{P}_2(H) := \mathcal{P}'(H) - \mathcal{P}_1(H).$$

For $\Gamma \in \mathcal{P} := \mathcal{P}(\Omega)$, set $M(\Gamma) := N_G(\Gamma)$ and $D(\Gamma) := K_+(\Gamma)$, except when Γ has blocks of size 2, where we set $D(\Gamma) := \hat{G}_\Gamma$.

6.2. *The following hold:*

(1) *$\mathcal{P}(H)$ is a sublattice of $\mathcal{P}$.*

(2) *Pick $\omega \in \Omega$, and for $U \in \mathcal{O}_H(H_\omega)$, set $\gamma(U) := \omega U$. Then the map*

$$U \mapsto \gamma(U)^H$$

is an isomorphism of the dual of the poset $\mathcal{O}_H(H_\omega)$ with the poset $\mathcal{P}(H)$.

Proof. If H acts on Γ and Σ in $\mathcal{P}(\Omega)$, then it acts on $\Gamma \vee \Sigma$ and $\Gamma \wedge \Sigma$, so (1) holds. See 5.18 in [1] for (2). □

6.3. *Suppose $\Gamma \in \mathcal{P}'(H)$. Then:*

(1) *$\Gamma \in \mathcal{P}_2$ if and only if $H_* \not\le D(\Gamma)$ if and only if $H_* \not\le \bar{D}(\Gamma)$.*

(2) *If Γ has blocks of size $k \le 4$, then $\Gamma \in \mathcal{P}_2(H)$.*

Proof. If $\Gamma \in \mathcal{P}_2$, then $H_* \not\le K = G_\Gamma$, so as $D := D(\Gamma) \le K$, $H_* \not\le D$. On the other hand, if $H_* \not\le D$, then $H_* \not\le K$ as K/D is solvable and $H_* = H_*^\infty$. Now (1) follows from 2.12.

Under the hypothesis of (2), K is solvable, so $H_* \not\le K$. Thus (2) holds. □

6.4. *Suppose $\Gamma \in \mathcal{P}_2(H)$. Then:*

(1) *$\hat{G}_\Gamma J$ and $\bar{D}(\Gamma)$ are in $\mathcal{W}_0$.*

(2) *If $\Gamma \le \Sigma \in \mathcal{P}'(H)$, then $\Sigma \in \mathcal{P}_2(H)$, $\hat{G}_\Sigma \le \hat{G}_\Gamma$, and if $|\Sigma| \ne n/2$, then we have $D(\Sigma) \le D(\Gamma)$.*

(3) *Let* $Y \in \mathcal{O}_G(D(\Gamma)H) - \{G\}$. *Then* Y *is imprimitive on* Ω, $\mathcal{P}'(Y)$ *has a greatest member* Γ_Y, $\Gamma_Y \leq \Gamma$, $D(\Gamma_Y) \leq Y$, *and* $D(\Gamma) \leq \hat{G}_{\Gamma_Y}$.

(4) *If* $D(\Gamma) \leq Y = WH$ *for some* $W \in \mathcal{W}_0$, *then we have* $D(\Gamma) \leq W \geq D(\Gamma_Y)$, *so* $\Gamma_Y \in \mathcal{P}_2(H)$.

Proof. Part (1) follows from 2.12. Assume $\Gamma \leq \Sigma$. Then by 3.4.2 in [8], $S_\Sigma \leq S_\Gamma$, so (2) follows.

Let k be the size of a block in Γ. We next prove (3). If $k > 2$, then (3) follows from 3.2 in [8]. Thus we may take $k = 2$. As H is not solvable, neither is $M(\Gamma)$, so $n > 8$. Hence by 4.5 in [8], Y is imprimitive on Ω. By 4.4.3 in [8], $\Sigma \leq \Gamma$ for each $\Sigma \in \mathcal{P}'(Y)$, so $D(\Gamma) \leq \hat{G}_\Sigma$. Thus as $\mathcal{P}(Y)$ is a sublattice of $\mathcal{P}$, $\mathcal{P}'(Y)$ has a greatest member Γ_Y. Let $\sigma \in \Sigma := \Gamma_Y$. By maximality of Σ, Y^σ is primitive. Further $D(\Gamma)^\sigma = \mathrm{Sym}(\sigma)_{\Gamma_\sigma}$, so by 3.2 in [8], $Y^\sigma = \mathrm{Sym}(\sigma)$. Then

$$\mathrm{Alt}(\sigma) = X^\sigma, \quad \text{where } X := \langle D(\Gamma)_{\Omega-\sigma}^{N_Y(\sigma)} \rangle \leq Y_{\Omega-\sigma},$$

so $D(\Sigma) \leq Y$. Hence (3) also holds when $k = 2$.

Finally assume the hypothesis of (4), and let $\Sigma \in \{\Gamma, \Gamma_Y\}$, r the size of a block of Σ, and $E := D(\Sigma)$. If $\Sigma = \Gamma$, then $E \leq Y$ by hypothesis, while if $\Sigma = \Gamma_Y$, then $E \leq Y$ by part (3). If $r \leq 4$, then $\Sigma \in \mathcal{P}_2(H)$ by 6.3.2, and E is solvable so $WE \in \mathcal{W}_0$ by 2.13. Hence $H_* \not\leq WE$ by 2.12, so as $H_*W/W = F^*(Y/W)$, it follows that $E \leq W$, and (4) holds in this case.

Thus we may take $r > 4$. Then E is a minimal normal subgroup of $EH = Y$, so either $E \leq W$ or $E \cap W = 1$. As $F^*(Y/W) = H_*W/W$ is simple, but E is not, the latter is impossible, so $E \leq W$. Thus $H_* \not\leq E$, so $\Sigma \in \mathcal{P}_2(H)$ by 6.3.1, completing the proof of (4). □

6.5. *Suppose* $\mathcal{P}_1(H) \neq \varnothing$. *Then:*

(1) $\mathcal{P}_1(H)$ *has a greatest member* $\Gamma_1(H)$.

(2) $\Gamma_1(H)$ *is the set of orbits of* H_* *on* Ω.

(3) *If* $\Sigma \in \mathcal{P}'(H)$ *with* $\Sigma \leq \Gamma_1(H)$, *then* $\Sigma \in \mathcal{P}_1(H)$.

(4) *Let* $W \in \mathcal{W}_0$, $Y := WH$, *and suppose* $\Sigma \in \mathcal{P}_1(H) \cap \mathcal{P}(Y)$ *such that* $W_\Sigma \neq 1$. *Let* $\sigma \in \Sigma$. *Then* Y^σ *is not almost simple.*

Proof. Let $\mathcal{O}$ be the set of orbits of H_* and $\Gamma \in \mathcal{P}_1(H)$. As H is transitive on Ω and $H_* \trianglelefteq H$, it follows that $\mathcal{O} \in \mathcal{P}(H)$. Also $H_* \leq \hat{G}_\Gamma$, so $\Gamma \leq \mathcal{O}$, and in particular $\mathcal{O} \in \mathcal{P}'(H)$. By construction, $H_* \leq G_{\mathcal{O}}$, so $\mathcal{O} \in \mathcal{P}_1$. Thus (1) and (2) hold.

Assume the hypothesis of (3). Then $\Sigma \leq \mathcal{O}$, so each orbit of H_* is contained in a block of Σ, and hence $H_* \leq G_\Sigma$, so (3) follows.

Assume the hypothesis of (4) with $\Sigma = \{\sigma_1, \dots, \sigma_m\}$ and $\sigma = \sigma_1$. Then

$$D := D(\Sigma) = D_1 \times \cdots \times D_m,$$

where $D_i = A_{\Omega - \sigma_i}$. Let $\pi_i : D \to D_i$ be the ith projection. Assume Y^σ is almost simple and set $X := Y_\Sigma^\infty$. As $\Sigma \in \mathcal{P}_1(H)$, $H_* \leq X$ and $X\pi_i = X^{\sigma_i} = F^*(Y^{\sigma_i})$ is a nonabelian simple group for $1 \leq i \leq m$. Thus by 1.4 in [7], there is $\Gamma \leq \Sigma$ such that

$$X = \prod_{\gamma \in \Gamma} X_\gamma,$$

where X_γ is a full diagonal subgroup of $D_\gamma = \prod_{i \in \gamma} X\pi_i$. Hence $\{X_\gamma : \gamma \in \Gamma\}$ is the set of components of X, and as H is transitive on Γ, X is a minimal normal subgroup of HX. But $W_\Sigma \neq 1$, so $1 \neq W_\Sigma^\sigma \trianglelefteq Y^\sigma$, and hence

$$X^\sigma = F^*(Y^\sigma) \leq W_\Sigma^\sigma,$$

so that $W_\Sigma^\infty \neq 1$. Then W_Σ^∞ is a nontrivial normal subgroup of HX contained in X, so $X \leq W$ by minimality of X. But now $H_* \leq X \leq W$, a contradiction. □

6.6. *Set $\mathcal{P}_J(H) := \{\Gamma \in \mathcal{P}'(H) : D(\Gamma) \leq J\}$. Assume $\mathcal{P}_J(H) \neq \varnothing$. Then:*

(1) *$\mathcal{P}_J(H)$ has a unique member $\Gamma_J(H)$.*

(2) *$\Gamma_J(H)$ is the greatest member of $\mathcal{P}'(H)$.*

(3) *Each $M \in \mathcal{O}_G(H)'$ is transitive and imprimitive on Ω, $\mathcal{P}'(M)$ has a greatest member Γ_M, $\Gamma_M \leq \Gamma_J$, and $D(\Gamma_J) \leq \hat{G}_{\Gamma_M}$.*

(4) *$\Gamma_J(H) \in \mathcal{P}_2(H)$.*

Proof. Let $M \in \mathcal{O}_G(H)'$, $\Gamma_0 \in \mathcal{P}_J(H)$, and $D_0 := D(\Gamma_0)$. As $H_* \not\leq J$, we get $H_* \not\leq D(\Gamma_0)$, so $\Gamma_0 \in \mathcal{P}_2(H)$ by 6.3.1. By definition, $D_0 \leq J \leq H \leq M$, so by 6.4.3, (3) holds with Γ_J replaced by Γ_0. Then specializing to the case $\Gamma \in \mathcal{P}'(H)$ and $M = N_G(\Gamma)$, we conclude Γ_0 is the greatest member of $\mathcal{P}'(H)$. As this holds for each $\Gamma_0 \in \mathcal{P}_J$, part (1) follows with $\Gamma_0 = \Gamma_J$, and then (2)–(4) follow from earlier remarks which showed these statements hold for Γ_0. □

Notation 6.7. Suppose $\mathcal{P}_J(H) \neq \varnothing$. In that event set

$$\Gamma_J := \Gamma_J(H), \quad M_J := M(\Gamma_J), \quad \text{and} \quad M_J^* := M_J / G_{\Gamma_J}.$$

For $\Sigma \in \mathcal{P}'(H)$, we define Σ^* to be the partition of Γ_J with blocks $\sigma^* = (\Gamma_J)_\sigma$ for $\sigma \in \Sigma$. By 6.6.2, $\Sigma \leq \Gamma_J$, so this makes sense. Define $\mathcal{P}'(H^*)$ to be the set of nontrivial H^*-invariant partitions of Γ_J. In the other direction, for $\Xi \in \mathcal{P}'(H^*)$, define $\varphi(\Xi) \in \mathcal{P}'(H)$ to be the partition with blocks

$$\phi(\xi) = \bigcup_{\alpha \in \xi} \alpha \quad \text{for } \xi \in \Xi.$$

For $W \in \mathcal{W}_0$, define $\Gamma_W := \Gamma_{WH}$ using 6.6.3, and set

$$M_W := M(\Gamma_W) \quad \text{and} \quad D_W := D(\Gamma_W).$$

6.8. *Assume $\mathcal{P}_J(H) \neq \varnothing$, and let $\mathcal{P}_*(H)$ denote the set of maximal members of $\mathcal{P}_2(H) - \{\Gamma_J\}$. Then:*

(1) *The map $\varphi : \mathcal{P}'(H^*) \to \mathcal{P}'(H) - \{\Gamma_J\}$ is an isomorphism of posets with inverse $\Sigma \mapsto \Sigma^*$.*

(2) *For $W \in \mathcal{W}_0$, $\Gamma_W \leq \Gamma_J$, $D_W \leq W$, and $\Gamma_W \in \mathcal{P}_2(H)$.*

(3) *$\mathcal{P}_*(H)$ has a unique member $\Gamma_* = \Gamma_*(H)$.*

(4) *Γ_*^* is the greatest member of $\mathcal{P}'(H^*)$.*

(5) *$U := \bar{D}(\Gamma_*) \in \mathcal{W}_0^*$ and $N_U(\Gamma_J) = J$.*

(6) *For $\gamma \in \Gamma_*$, define $\mathfrak{J}_\gamma := J \cap A_{\Omega-\gamma}$. Then $\mathfrak{J}_\gamma = N_{A_{\Omega-\gamma}}(\gamma^*)$.*

(7) $J^*_{\Gamma_J-\gamma} = \mathrm{Sym}(\Gamma_J)_{\Gamma_J-\gamma^*}$.

(8) *Each member of $\mathcal{O}_{M_J^*}(H^*)'$ is imprimitive on Γ_J.*

(9) *Γ_* is the unique $\Sigma \in \mathcal{P}'(H)$ such that $\bar{D}(\Sigma) \in \mathcal{W}_0^*$.*

Proof. Part (1) follows from 6.6.2. By 6.6.4, $\Gamma_J \in \mathcal{P}_2(H)$. Further $D_J \leq J \leq H$, so for $W \in \mathcal{W}_0$, $D_J H \leq WH$. Thus (2) follows from parts (3) and (4) of 6.4 applied to Γ_J in the role of "Γ".

Let $\mathcal{W}_1 := \{W \in \mathcal{W}_0 : \Gamma_W \neq \Gamma_J\}$. By 3.3, $\mathcal{W}_1 \neq \varnothing$. Pick $U \in \mathcal{W}_1$. By (2), $D_U \leq U$, $\Gamma_U \in \mathcal{P}_2(H)$, and $\Gamma_U \leq \Gamma_J$, so as $\Gamma_U \neq \Gamma_J$, $\Gamma_U < \Gamma_J$. If $D_U \leq J$, then $\Gamma_U = \Gamma_J$ by 6.6.1, which we just saw is not the case. Therefore $D_U \not\leq J$. Also we have $\mathcal{P}_*(H) \neq \varnothing$, so taking $U := \bar{D}(\Gamma)$ for $\Gamma \in \mathcal{P}_*(H)$, we may assume $\Gamma_U = \Gamma \in \mathcal{P}_*(H)$, and we have shown $U \not\leq J$.

Suppose $V \in \mathcal{W}_0^*$ and $V < U$. Then by definition in 6.7, Γ_V is the unique maximal member of $\mathcal{P}'(HV)$, so as $HV \leq HU$, we conclude that $\Gamma \leq \Gamma_V$. Hence as $\Gamma_* \in \mathcal{P}_*(H)$, we get $\Gamma_V = \Gamma$ or Γ_J by maximality of Γ. In the former case, $D(\Gamma) = D_V \leq V$ by (2), so $U = \bar{D}(\Gamma) \leq V$, contradicting $V < U$. Thus we have $V \leq N_U(\Gamma_J)$, so $N_U(\Gamma_J)$ is the unique maximal member of $\{W \in \mathcal{W}_0 : W < U\}$, contrary to 3.4.

Thus no such V exists, so $U \in \mathcal{W}_0^*$. Set $E := N_U(\Gamma_J)$. Now E is H-invariant, so $\bar{E} \in \mathcal{W}_0$ by 2.7. Then as $U \in \mathcal{W}_0^*$, $\bar{E} = U$ or J. As $\Gamma_J \neq \Gamma$, $\bar{E} \neq U$, so $E \leq J$.

However $D(\Gamma) = D_U \leq U$ and $N_{D(\Gamma)}(\Gamma_J)$ is a maximal H-invariant subgroup of $D(\Gamma)$ by 3.7.2 in [8], using the fact that $D(\Gamma_J) \leq H$ when the blocks of Γ_J are of size 2, and those of Γ are of size 8. Therefore as $U = \bar{D}(\Gamma) \in \mathcal{W}_0^*$, it follows that $D(\Gamma) \cap J = N_{D(\Gamma)}(\Gamma_J)$ and $J = N_U(\Gamma_J)$. This establishes (5), once we prove (3). Further as $J = N_U(\Gamma_J)$ and $D(\Gamma) \leq U$, (6) follows.

Next let $\alpha, \beta \in \Gamma_J$ with $\alpha, \beta \subseteq \gamma \in \Gamma$, and set $m := |\alpha|$. We next prove (7). By (6), it suffices to show there is a transposition $t \in M_J^*$ with cycle (α, β) on Γ_J, such that $t = s^*$ for some $s \in A_{\Omega-(\alpha\cup\beta)}$. If m is even, we can choose s so that s^* is of cycle type 2^m. Thus we may assume m is odd. Now $m - 1 = 2^a k$ with k odd and $a > 0$. For $\delta \in \{\alpha, \beta\}$, let $t_\delta \in \mathrm{Sym}(\Gamma_J)$ have one fixed point and k cycles of length 2^a, with $\mathrm{Mov}(t_\delta) \subseteq \delta$. Let $t \in \mathrm{Sym}(\Gamma_J)$ interchange α and β with $t^2 = t_\alpha t_\beta$ and $\mathrm{Mov}(t) = \alpha \cup \beta$. Then t has one cycle of length 2 and k cycles of length 2^{a+1}, so t is even and hence $t \in M_J^*$. Thus (7) holds.

Now (7) and 3.2.1 in [8] applied to M_J^* acting on Γ_J imply (8), and, together with 3.2.2 in [8], say that Γ^* is the greatest member of $\mathcal{P}'(H^*)$. As Γ can be chosen to be any member of $\mathcal{P}_*(H)$, (3) and (4) follow from this fact and (1).

Let $\mathcal{S}$ consist of those $\Sigma \in \mathcal{P}'(H)$ such that $\bar{D}(\Sigma) \in \mathcal{W}_0^*$. By (5), $\Gamma_* \in \mathcal{S}$. Conversely suppose $\Sigma \in \mathcal{S}$ and set $V := \bar{D}(\Sigma)$. Then $\Sigma \in \mathcal{P}_2(H)$ and $\Sigma \leq \Gamma_*$ by (1) and (4). As $\Gamma_* < \Gamma_J$, $|\Gamma_*| < n/2$, so $D(\Gamma_*) \leq D(\Sigma)$ by 6.4.2. But now $U = \bar{D}(\Gamma_*) \leq \bar{D}(\Sigma) = V$, so as $V \in \mathcal{W}_0^*$, $U = V$, and hence $\Sigma = \Gamma_*$. Thus (9) is established. □

6.9. $\mathcal{P}_J(H) = \varnothing$.

Proof. Assume otherwise and adopt Notation 6.7 together with the notation in 6.8. Set $U := \bar{D}(\Gamma_*)$. Thus $U \in \mathcal{W}_0^*$ by 6.8.5. Let $\mathcal{C}$ be the connected component of Λ containing U and $Y := \hat{G}_{\Gamma_J}$. As Λ is disconnected there exists $V \in \mathcal{W}_0^*$ with $V \notin \mathcal{C}$. By 6.8.9, $V \neq \bar{D}_V$, while $\bar{D}_V \in \mathcal{W}_0$ is contained in V by 6.8.2. Thus we have $D_V \leq J$, so by 6.6.1, $\Gamma_V = \Gamma_J$, so $HV \leq N_G(\Gamma_J) = M_J$. Next by 6.8.7,

$$(M_J^*)_{\Gamma_*^*} \leq J^* \leq V^*,$$

so by 6.8.7 and 3.2 in [8], either $\mathrm{Alt}(\Gamma_J) \leq V^*$ or $\mathcal{P}'(H^*V^*)$ has a greatest member Σ^*, $\Sigma^* \leq \Gamma_*^*$, and $(M_J^*)_{\Sigma^*} \leq V^*$. As $Y \leq J$ and $H_* \not\leq V$, the latter holds.

Next $H^* \cap (M_J^*)_{\Sigma^*} \leq H^* \cap V^* = (H \cap V)^* = J^*$ as $Y \leq J$. Thus the kernel of the action of H_* on Σ^* is contained in J, so from 6.8.1, the kernel of the action of H_* on $\Sigma := \varphi(\Sigma^*)$ is also contained in J, and hence $\Sigma \in \mathcal{P}_2$ by 6.3.1, so we obtain $E := \bar{D}(\Sigma) \in \mathcal{W}_0$ by 6.4.1. By 6.8.4, $\Sigma \leq \Gamma_*$. As we saw during the proof of 6.8.9, $|\Gamma_*| > n/2$, so by 6.4.2, $D(\Gamma_*) \leq D(\Sigma)$, and hence $U = \bar{D}(\Gamma_*) \leq E$. Moreover $D(\Sigma) \cap M_J =: E_J$ is H-invariant, so $\bar{E}_J \in \mathcal{W}_0$. Also

$$E_J^* \leq (M_J^*)_{\Sigma^*} \leq V^*,$$

so $\bar{E}_J \le V$. If $\Sigma \ne \Gamma_*$, then E_J does not act on $(M_J^*)_{\Gamma_*^*} \trianglelefteq J^*$, so $E_J \not\le J$. But then $U, E, \bar{E}_J, V$ is a path in Λ', contradicting $V \notin \mathcal{C}$. Hence $\Sigma = \Gamma_*$, so V acts on $D(\Gamma_*)$. Then by 3.9.3 in [6], $UV = H_*V \le M_J$, which is impossible as D_U does not act on D_J. This completes the proof of the lemma. □

7 $\mathcal{P}_2(H) \ne \varnothing$

In this section we assume:

Hypothesis 7.1. Hypothesis 6.1 holds and $\mathcal{P}_2(H) \ne \varnothing$.

Notation 7.2. For $\Sigma \in \mathcal{P}_2(H)$, write $\Sigma = \{\sigma_1, \dots, \sigma_m\}$, where $m = |\Sigma|$, and set $k(\Sigma) = |\sigma_i|$, the size of the blocks in Σ. If $k(\Sigma) \ne 2$, by definition of $D = D(\Sigma)$, $D = D_1 \times \cdots \times D_m$, where $D_i = A_{\Omega - \sigma_i}$ acts faithfully as the alternating group $A_{k(\Sigma)}$ on σ_i.

7.3. *Let $\Sigma \in \mathcal{P}_2(H)$. Then:*

(1) $\bar{D}(\Sigma) \in \mathcal{W}_0'$.

(2) $|\Sigma| \ge 5$.

Proof. By 6.4.1, $U := \bar{D}(\Sigma) \in \mathcal{W}_0$. By 6.9, we have $\Sigma \notin \mathcal{P}_J(H)$, so $D(\Sigma) \not\le J$, and hence $U \in \mathcal{W}_0'$. That is (1) holds.

As $\Sigma \in \mathcal{P}_2(H)$, $H_* \not\le G_\Sigma$, so $H_*^\Sigma \ne 1$. Then as $H_* = H_*^\infty$, (2) holds. □

7.4. *Suppose $\Sigma \in \mathcal{P}_2(H)$ with $k(\Sigma) \ne 2$. Let $D := D(\Sigma)$ and $K := \hat{G}_\Sigma$. Then:*

(1) $\bar{K} \in \mathcal{W}_0'$.

(2) *$K = DX$, where $\bar{X} \in \mathcal{W}_0$ with $\bar{D} \cap \bar{X} = J$ and $\bar{X} \cap K = X$.*

(3) *$\mathcal{W}_0(\le \bar{D}) = \{\bar{B} : B \in \mathcal{V}_D(H)\}$ and $\mathcal{W}_0(\le \bar{K}) = \{\bar{B} : B \in \mathcal{V}_K(H)\}$.*

(4) *Each member of $\mathcal{V}_D(H)$ is XH-invariant.*

(5) *In the notation of 2.5, $\Xi_{KH} \cong \mathcal{V}_K(H) \cong \Delta(r)$, $\Xi_{DH} \cong \mathcal{V}_D(H) \cong \Delta(s)$, and $\Xi_{XH} \cong \mathcal{V}_X(H) \cong \Delta(r-s)$ for some $1 \le s \le r$.*

Proof. By 6.4.1, $\bar{K} \in \mathcal{W}_0$, so (1) follows from 7.3.1.

We next prove (2), by applying 3.7 to D, KH in the roles of Y, X, and with $\mathcal{C}$ the connected component of $\bar{D}$. Then, in the notation of 3.7, we have $\bar{K} = L_{\mathcal{D}}$ and $\bar{D} = L_{\mathcal{A}}$. Set $V := L_{\mathcal{B}}$ and $X := V \cap K$. Then by construction,

$$V \cap \bar{D} = L_{\mathcal{B}} \cap L_{\mathcal{A}} = L_{\mathcal{B} \cap \mathcal{A}} = L_{\varnothing} = J,$$

and as $D \trianglelefteq \bar{K}$,

$$\bar{K} = \mathcal{L}_{\mathcal{D}} = \langle \mathcal{L}_{\mathcal{A}}, \mathcal{L}_{\mathcal{B}} \rangle = DV,$$

so $K = K \cap DV = D(K \cap V) = DX$. Finally $\bar{K} = KJ = DXJ = D\bar{X}$ so

$$V = V \cap \bar{K} = V \cap D\bar{X} = (V \cap D)\bar{X} = J\bar{X} = \bar{X},$$

completing the proof of (2).

Moreover as $\bar{K} \in \mathcal{W}_0'$, (3) follows from 3.7.3. Then 3.7.4 and (3) imply (4). Next 2.5 says

$$\Xi_{KH} \cong \mathcal{W}_0(\leq \bar{K}) \cong \Delta(r)$$

for some $r \geq 1$, and indeed $\mathcal{D} = \mathcal{W}_0^* \cap \bar{K}$, so $r = |\mathcal{D}|$. Similarly

$$\Xi_{KH} \cong \mathcal{W}_0(\leq \bar{D}) \cong \Delta(s)$$

with $\mathcal{A} = \mathcal{W}_0^* \cap \bar{D}$ and $s = |\mathcal{A}|$, so $\Xi_{XK} \cong \Delta(t)$ with $t = |\mathcal{B}| = r - s$, establishing (5). $\square$

7.5. *Assume $\Sigma \in \mathcal{P}_2(H)$ with $k(\Sigma) \geq 5$. Define $K := \hat{G}_\Sigma$ and X as in* 7.4. *Then for each $V \in \mathcal{V} = \mathcal{W}_0(\leq \bar{D})$ and each $1 \leq i \leq m := |\Sigma|$,*

(1) *$V \cap D = \prod_i V_i$, where $1 \neq V_i = V \cap D_i$.*

(2) *$V \cap D$ is XH-invariant.*

(3) *$N_H(\sigma_i)^{\sigma_i}$ is primitive.*

(4) *Σ is maximal in $\mathcal{P}'(H)$.*

(5) *J_i is transitive on σ_i.*

Proof. Adopt the notation of 7.4 and its proof. Then $\bar{D} = L_{\mathcal{A}}$, $\bar{X} = L_{\mathcal{B}}$, and by 3.7.4, for each $\alpha \subseteq \mathcal{A}$, $D_\alpha = D \cap L_\alpha$ is XH-invariant. Pick α to be maximal proper subset of $\mathcal{A}$, and set $M := HD_\alpha$ and $Y := HD$. Then by 7.4.3, M is maximal in Y. Further $\bar{D}_\alpha = V \in \mathcal{V}$.

Let $B := HK$, $F := HX$, $B^i := N_B(\sigma_i)^{\sigma_i}$, and for $P \leq B$, $P^i := N_P(\sigma_i)^{\sigma_i}$. Then $B^i = \mathrm{Sym}(\sigma_i)$, $D^i = D_i^i = \mathrm{Alt}(\sigma_i)$, and $X^i \not\leq \mathrm{Alt}(\sigma_i)$, so $F^i \not\leq \mathrm{Alt}(\sigma_i)$.

As $k = k(\Sigma) \geq 5$, $D_i \cong A_k$ is a nonabelian simple group from 7.2. Thus we have $F^*(Y) = D$ and $H \leq M$, so M is transitive on the set $\{D_1, \dots, D_m\}$ of components of Y. Thus, in the language of [4], Y is faithful and primitive on Y/M, and is complemented, diagonal, or semisimple. In the last case,

$$M \cap D = M_1 \times \cdots \times M_m,$$

where $M_i = M \cap D_i \neq 1$. Then as

$$M \cap D = HD_\alpha \cap D = (H \cap D)D_\alpha = (J \cap D)D_\alpha = D_\alpha,$$

we have $V = \prod_i V_i$. In particular if Y on Y/HD_α is semisimple for each maximal α, then (1) holds, modulo the assertion that $W_i \neq 1$ for each $W \in \mathcal{V}$, which we now prove.

Suppose $W_i = 1$. Then $W \cap D = 1$, so $W = J$. Now $H^i \cap D^i$ is H^i-invariant, so $H^i \cap D^i = U_i^i$ for some $U = L_\gamma$ and $\gamma \subset \mathcal{A}$ by 7.4.3. As H^i acts on Z_i for each $Z \in \mathcal{V}$, also U_i acts on Z_i. Claim $\gamma = \varnothing$, so that $U = J$ and hence $U_i^i = J_i^i = 1$. For if not, $\delta = \mathcal{A} - \gamma \neq \varnothing$ or $\mathcal{A}$, so by 3.7.3, $Z := L_\delta \neq J$ or D, and U_i acts on Z_i. Then by 3.7.3, $D = \langle U_i, Z_i \rangle$ acts on $Z_i \neq 1$ or D, a contradiction. This establishes the claim, and hence $H^i \cap D^i = 1$. Therefore $|H^i| \leq 2$. But then as $\mathcal{O}_{D_i}(N_H(\sigma_i)) = \{W_i : W \in \mathcal{A}\} \cong \Delta(s)$ by 7.4.5, we have a contradiction.

Thus we may assume Y is complemented or diagonal. But by 7.4.2, $K = DX$ and D_α and $D \cap J$ are HX-invariant. Now by 7.4.2, $X \cap D \leq J$, while

$$\mathrm{Aut}_X(D_i) \not\leq \mathrm{Inn}(D_i)$$

by paragraph two of the proof. Thus if Y is complemented, then $D \cap J = D_\alpha = 1$, so

$$\mathrm{Aut}_X(D_i) \cong \boldsymbol{Z}_2 \quad \text{and} \quad \mathrm{Aut}_M(D_i) = \mathrm{Aut}_H(D_i).$$

Then as $\mathrm{Aut}_H(D_i)$ acts on $\mathrm{Aut}_X(D_i)$, $\mathrm{Inn}(D_i) \not\leq \mathrm{Aut}_M(D_i)$, contradicting Y complemented.

This leaves the case Y diagonal. In this case $D_\alpha = \prod_\rho D_{\alpha,\rho}$, where $D_{\alpha,\rho}$ is a full diagonal subgroup of D_ρ, ρ varies over the blocks of an H-invariant partition of $\{1, \ldots, m\}$, and $D_\rho = \prod_{i \in \rho} D_i$. As $K = DX$, D_α is X-invariant, and $m > 2$ by 7.3.2, this too is a contradiction. So (1) is established.

Part (2) follows from 7.4.4.

Suppose H^i is not primitive. Then $H^i \leq P^i$ the stabilizer of some nontrivial partition Γ of σ_i. Let $P_i := N_{D_i}(\Gamma)$ and $P := \langle P_i^H \rangle$. Then $P \in \mathcal{V}_D(H)$, so by 7.4.4, X acts on P. Therefore F^i acts on P_i, and hence F^i is also imprimitive. Therefore by 7.4.5 and 5.2 in [8], $\mathcal{O}_{B^i}(F^i)$ is the lattice with maximal members $\{B_1^i, \ldots, B_s^i\}$, where B_j^i is the stabilizer of a nontrivial partition Γ_j of σ_i, and $\Gamma_1 < \cdots < \Gamma_s$. Then setting $P_{i,j} = N_{D_i}(\Gamma_j)$ and $P_j = \langle P_{i,j}^H \rangle$, it follows from 7.4 that the maximal members of $\mathcal{W}_0(< \bar{D})$ are $\bar{P}_j$, $1 \leq j \leq s$. Let Σ_j be the union of the images of the blocks in Γ_j under H. Then $\Sigma_j \in \mathcal{P}'(H)$ with $\Sigma < \Sigma_j$. Further

$$J = \bigcap_j \bar{P}_j$$

contains $D(\Sigma_s)$. But then we get $\Sigma_s \in \mathcal{P}_J(H)$, contrary to 6.9. This completes the proof of (3), and (3) implies (4). Finally as J_i is $N_H(\sigma_i)$-invariant, (1) and (3) imply (5). □

7.6. *Assume $\Sigma \in \mathcal{P}_2(H)$ and $k(\Sigma) = 4$. Let $D := D(\Sigma)$ and $U := \bar{D}$. Then:*

(1) *$O_2(D) = D \cap J$ and $U \in \mathcal{W}_0^*$.*

(2) *For each $W \in \mathcal{W}_0$, WH is imprimitive.*

Proof. Let $Q := O_2(D)$, $\sigma := \sigma_1$, and assume $Q \not\leq J$. By 7.3.1, $U \in \mathcal{W}_0'$, so $\bar{Q} \in \mathcal{W}_0'$ by 2.7.1. Further either $D = Q(J \cap D)$ or $\bar{Q} < U$. In any event by 3.7, there exists $Y \leq D$ such that $D = QY$, $\bar{Y} \in \mathcal{W}_0$, and $\bar{Q} \cap \bar{Y} = J$. Thus

$$Y \cap Q = Q \cap J.$$

Let $M := YH$. Then M is transitive on Σ and Y is irreducible on $Q_i := Q \cap D_i$, so M is irreducible on Q. Thus as $Q \not\leq J$ and $Q \cap J = Y \cap Q \trianglelefteq M$, we conclude that $Q \cap J = 1$. Thus as $Q \cap J = Y \cap Q$ and $D = QY$, Y is a complement to Q in D. That is $Y \in \mathrm{Syl}_3(D)$. Therefore

$$Y_1 := Y \cap D_1 \in \mathrm{Syl}_3(D_1).$$

But $Y = \bar{Y} \cap D$ and $\bar{Y}$ is H-invariant, so $N_H(\sigma)$ acts on Y_1. This is impossible as Y_1 fixes a unique point of σ, and H is transitive on Ω, so $N_H(\sigma)$ is transitive on σ.

Therefore $Q \leq J$. Now define K and X as in 7.4, and let $M := XH$. Then $K = DX$ and by 7.4.2,

$$X \cap D = J \cap D,$$

so $Q \leq X$. Set $D^* = D/Q$. Then $D_1^* = [D_1^*, X^*]$ and H is transitive on Σ, so M is irreducible on D^*. Thus either $Q = X \cap D$ or $D \leq X$, and the latter is impossible as $X \cap D = J \cap D$ and $D \not\leq J$ by 7.3.1. Hence $Q = \bar{X} \cap D$, so as $Q \leq J \leq \bar{X}$, $Q = J \cap D$. Further as X is irreducible on D^* and $Q \leq J$, $U \in \mathcal{W}_0^*$. This completes the proof of (1).

Next $E := Q_{\Omega - \sigma} \cong E_4$ and for each $e \in E^\#$, $\mathrm{Mov}(e) = \sigma$. Let $W \in \mathcal{W}_0$. Then $E \leq J \leq W$, and by 2.14, WH is not almost simple. Therefore by 4.2 in [8], WH is imprimitive, establishing (2). □

7.7. *For* $\Sigma \in \mathcal{P}_2(H)$, $k(\Sigma) \neq 3$.

Proof. Assume $\Sigma \in \mathcal{P}_2(H)$ with $k(\Sigma) = 3$, let $D := D(\Sigma)$, $U := \bar{D}$, and adopt the notation of 7.4. Set $M := HX$ and $Y := HK$. Then $N_M(D_1)$ is irreducible on D_1 and H is transitive on Σ, so M is irreducible on D. But, using 7.4.2,

$$M \cap K = HX \cap K = X(H \cap K) = X(J \cap K) = X,$$

so $M \cap D = X \cap D$. Thus as M is irreducible on D, $X \cap D = 1$ or D. By 7.4.2, $X \cap D = J \cap D$, so as $D \not\leq J$ by 7.3.1, $D \cap X = 1$. By construction in 7.4.2, $K = DX$, so X is a complement to D in K. That is $X \in \mathrm{Syl}_2(K)$. Let $\sigma = \sigma_1$. Now $K^\sigma = \mathrm{Sym}(\sigma) \cong S_3$ and H acts on X, so H^σ acts on $X^\sigma \in \mathrm{Syl}_2(D^\sigma)$. This is impossible as X^σ fixes a unique point of σ, whereas H is transitive on Ω, so H^σ is transitive on σ. □

7.8. *Assume $\Sigma \in \mathcal{P}_2(H)$ and $k(\Sigma) = 2$. Let $D := D(\Sigma)$, $J_D := J \cap D$, $U := \bar{D}$, and $\mathcal{W}_0^* \cap U = \{U_1, \dots, U_r\}$. Then $J_D \trianglelefteq HD$, and setting $D^* = D/J_D$,*

$$D^* = X_1^* \oplus \cdots \oplus X_r^*,$$

where $X_i := U_i \cap D$, $X_i^* = [X_i^*, H]$, and for $i \neq j$, X_i^* and X_j^* are inequivalent $\boldsymbol{F}_2H$-submodules of D^*.

Proof. As H acts on the abelian group D and $J \trianglelefteq H$, $J_D \trianglelefteq HD$.

By 7.3.1, $U \in \mathcal{W}_0'$, so by 3.7.3,

$$D = U \cap D = \langle X_i : 1 \leq i \leq r \rangle,$$

and $\mathcal{X} = \{X_i^* : 1 \leq i \leq r\}$ is the set of irreducible $\boldsymbol{F}_2H$-submodules of D^*. By 3.5.2, $X_i^* = [X_i^*, H]$. If $i \neq j$, then by 3.7.3, $\mathcal{X} \cap (X_i^* + X_j^*) = \{X_i^*, X_j^*\}$, so X_i^* is not $\boldsymbol{F}_2H$-isomorphic to X_j^*, completing the proof. □

7.9. *Assume there exists $\Sigma \in \mathcal{P}_2(H)$ such that $k(\Sigma) \geq 3$. Then for each $W \in \mathcal{W}_0$, HW is imprimitive on Ω.*

Proof. Let $1 \neq W \in \mathcal{W}_0$ and $Y := HW$. By 2.14:

($*$) Y is not almost simple.

Choose $\Sigma \in \mathcal{P}_2(H)$ and set $D := D(\Sigma)$, $U := \bar{D}$, $k := k(\Sigma)$. We may assume $k > 2$ and Y is primitive, and it remains to derive a contradiction. By 7.6.2 and 7.7, $k > 4$. Set $\sigma := \sigma_1$ and $J_1 := J \cap D_1$. As $k > 4$, J_1 is transitive on σ by 7.5.5.

Next $J_1 \leq Y$ and Y is primitive on Ω, so by 3.5 in [8] and ($*$), $n = 2^{a+1}$ for some $a > 1$, Y is the stabilizer of an affine structure, and $J_1 \not\leq E := F^*(Y)$. As Y is the stabilizer of an affine structure, E is the unique minimal normal subgroup of Y, so $E \leq W$. Further $J_1 \leq J \leq W$, so as $J_1 \not\leq E$, $E < W$. However as Y is the stabilizer of an affine structure over $\boldsymbol{F}_2$ and $n > 4$, Y/E is simple, so $W = Y$. This is a contradiction as $F^*(Y/W) \cong L \neq 1$. □

7.10. *For each $\Sigma \in \mathcal{P}_2(H)$, $\bar{D}(\Sigma) \in \mathcal{W}_0^!$.*

Proof. Let $\Sigma \in \mathcal{P}_2(H)$, $D := D(\Sigma)$, and $U := \bar{D}$. Assume $U \notin \mathcal{W}_0^!$; we derive a contradiction.

Let $k := k(\Sigma)$, let $\{W_i : 1 \leq i \leq t\}$ be the members of $\mathcal{W}_0$ such that U is a maximal H-invariant subgroup of W_i, and set $Y_i := W_i H$. Then $D \leq W_i \leq Y_i$, so by parts (3) and (4) of 6.4:

(1) For $1 \leq i \leq t$, Y_i is imprimitive on Ω, $\mathcal{P}'(Y_i)$ has a greatest member Γ_i, $\Gamma_i \in \mathcal{P}_2$, and $\Gamma_i \leq \Sigma$.

Next by 3.8:

(2) For some $1 \leq j \leq t$, W_j does not act on D. Thus $\Gamma_j \neq \Sigma$, so $\Gamma_j < \Sigma$.

Let $\Gamma := \Gamma_j$ and $k' := k(\Gamma)$. If $k' > 4$, then Γ is maximal in $\mathcal{P}'(H)$ by 7.5.4, contradicting $\Gamma < \Sigma$. Similarly if $k' = 2$, then Γ is maximal in $\mathcal{P}'(H)$, so it follows from 7.7 that:

(3) $k' = 4$.

However as $k' = 4$, 7.6.1 says that $W_j \in \mathcal{W}_0^*$, contradicting $J < U < W_j$. This completes the proof of the lemma. □

7.11. *For* $\Sigma \in \mathcal{P}_2(H)$, $k(\Sigma) \neq 4$.

Proof. Assume $\Sigma \in \mathcal{P}_2(H)$ with $k(\Sigma) = 4$. Then by 7.6.1, $\bar{D}(\Sigma) \in \mathcal{W}_0^*$, contrary to 7.10. □

7.12. *Assume* $\Sigma \in \mathcal{P}_2(H)$ *with* $k(\Sigma) > 2$. *Then:*

(1) $k(\Sigma) > 4$.

(2) $\mathcal{P}_2(H) = \{\Sigma\}$.

(3) *For each* $\Gamma \in \mathcal{P}'(H)$, $\Gamma \leq \Sigma$.

Proof. By 7.7 and 7.11, $k := k(\Sigma) > 4$. Adopt the notation of 7.5 and set $\sigma := \sigma_1$. By 7.5.5, J_1^σ is transitive. Let $\Gamma \in \mathcal{P}'(H)$ and $\gamma \in \Gamma$ with $\sigma \cap \gamma \neq \varnothing$. As J_1^σ is transitive, 1.1 says that $\sigma \subseteq \gamma$ or $\gamma \subseteq \sigma$. Thus as H is transitive on Γ and Σ, it follows that Γ and Σ are comparable. Further if $\Sigma \leq \Gamma$, then $\Sigma = \Gamma$ by 7.5.4. This establishes (3).

Suppose $\Gamma \in \mathcal{P}_2(H) - \{\Sigma\}$. Then by (3), $\Gamma < \Sigma$. By 6.4.2, $D < D(\Gamma)$, so by 7.10, $\bar{D} = \bar{D}(\Gamma)$. But then $D(\Gamma) = DJ \leq N_G(D)$, a contradiction. Hence (2) holds. □

7.13. *For each* $\Sigma \in \mathcal{P}_2(H)$, $k(\Sigma) = 2$.

Proof. Assume $\Sigma \in \mathcal{P}_2(H)$ with $k := k(\Sigma) \neq 2$. By 7.12, $k > 4$, so we adopt the notation of 7.5. By 7.5.5, $J_1 := J \cap D_1$ is transitive on $\sigma = \sigma_1 \in \Sigma$.

Let $\mathcal{C}$ be the connected component of Λ' containing $\bar{D}$. By 7.10, $\bar{D} \in \mathcal{W}_0^!$, so there exists a unique $U \in \mathcal{C} \cap \mathcal{W}_0^*$ with $U \not\leq \bar{D}$. By 7.9, $Y := UH$ is imprimitive. By 7.12.3, for each $\Gamma \in \mathcal{P}'(Y)$, $\Gamma \leq \Sigma$. Thus $\mathcal{P}'(Y)$ has a unique maximal member Γ. As $\Gamma \leq \Sigma$, $\sigma \subseteq \gamma \in \Gamma$. By 3.8.1, U does not act on D, so $\Gamma < \Sigma$. Then by 7.12.2, $\Gamma \in \mathcal{P}_1$, so $H_* \leq D(\Gamma)$.

As Γ is maximal in $\mathcal{P}'(Y)$, Y^γ is primitive. By 6.5.4, Y^γ is not almost simple. As $J_1^\gamma \leq Y^\gamma$ and J_1 is transitive on σ, it follows from 3.5 in [8] that $k(\Gamma) = 2k$,

$k = 2^a$, Y^γ is the stabilizer of an affine structure on γ, and $J_1^\gamma \not\leq F^*(Y^\gamma)$. As $J_1 \leq U$, it follows that

$$X = \langle J_1^{N_Y(\gamma)} \rangle \leq U \cap B(\gamma),$$

where $B(\gamma) = \hat{G}_{\Omega-\gamma}$, and that $F^*(Y^\gamma) \leq X^\gamma$. Therefore as $J_1^\gamma \not\leq F^*(Y^\gamma)$ and $Y^\gamma/F^*(Y^\gamma)$ is simple, $X^\gamma = Y^\gamma$. In particular as $H_* \leq D(\Gamma)$, $H_*^\gamma \leq X^\gamma$. Thus $H_* \leq \langle X^H \rangle \leq U$, a contradiction. This establishes the lemma. □

Notation 7.14. By Hypothesis 7.1, $\mathcal{P}_2(H) \neq \varnothing$. In the remainder of the section, let $\Sigma \in \mathcal{P}_2(H)$, $D := D(\Sigma)$, and $W =: \bar{D}$. By 7.13, we have $k(\Sigma) = 2$, so Σ is maximal in $\mathcal{P}'(H)$. By 7.10, $W \in \mathcal{W}^!$. Let $\mathcal{C}$ be the connected component of Λ' containing W. As $W \in \mathcal{W}^!$, there is a unique $U \in \mathcal{C} \cap \mathcal{W}_0^*$ such that $U \not\leq W$. Set $\mathcal{C}^! := \mathcal{C} \cap \mathcal{W}_0^!$, and set $s := |\mathcal{C}^! - \{W\}|$. Thus U is contained in each member of $\mathcal{C}^! - \{W\}$.

Let $\mathfrak{S}$ be the set of pairs (V, Γ) such that $V \in \mathcal{C}^! - \{W\}$ and $\Gamma \in \mathcal{P}'(VH)$. Let $\mathfrak{V}$ consist of those $V \in \mathcal{C}^! - \{W\}$ such that $m_2(V \cap D) \geq m_2(D)/2$. Let z be the fixed-point-free involution in S such that Σ is the set of orbits of $Z := \langle z \rangle$ on Ω.

7.15. $\mathcal{P}_2(UH) = \varnothing$.

Proof. Suppose $\Gamma \in \mathcal{P}_2(UH)$, and let $E := D(\Gamma)$. By 7.13, we have $k(\Gamma) = 2$ and $\bar{E} \in \mathcal{W}_0^!$ by 7.10.

As $k(\Gamma) = 2$, E is solvable. Then by 2.13 applied to $U, E, 1$ in the roles of W, X, Y, $UE \in \mathcal{W}_0$. Therefore as $\bar{E} \in \mathcal{W}_0^!$, $UE = \bar{E}$. Then $U \leq \bar{E}$, so $\bar{E} \in \mathcal{C}$. Then as $W, \bar{E} \in \mathcal{W}_0^!$, $W \cap \bar{E} =: V \in \mathcal{W}_0'$, and by 3.7, $E = (V \cap E)(U \cap E)$. As E is abelian and J acts on $U_E := U \cap E$ and $J_E = J \cap E$, $V = (V \cap E)J$ acts on U_E and J_E. From 7.8, H is irreducible on U_E/J_E, so as $V \cap D$ is an H-invariant 2-group, $V \cap D$ centralizes U_E/J_E. Thus $[U_E, V \cap D] \leq J_E \leq V \cap E$. Thus $E = (V \cap E)U_E$ acts on $Y = (V \cap E)(V \cap D)$. By symmetry, D acts on Y, so $Y \trianglelefteq M = \langle D, E, H \rangle$. However this contradicts 3.8.1. □

7.16. *For each* $(V, \Gamma) \in \mathfrak{S}$, $\Gamma \in \mathcal{P}_1$.

Proof. As $U \leq V$, this follows from 7.15. □

7.17. *Assume* $(V, \Gamma) \in \mathfrak{S}$ *such that* $\Gamma \not\leq \Sigma$. *Then:*

(1) $\Sigma \vee \Gamma = \infty$.

(2) $G_\Gamma \cap G_\Sigma = 1$.

(3) $[H_\Gamma, V \cap D] = 1$.

(4) H_* *centralizes* $V \cap D$.

(5) $m_2(V \cap D) \leq n/10$.

Proof. Part (1) follows from the maximality of Σ in $\mathscr{P}'(H)$. Then (2) follows from (1) and 1.2, and (2) implies (3). By 7.16, we have $\Gamma \in \mathscr{P}_1$, so $H_* \leq H_\Gamma$. Thus (3) implies (4).

As $k(\Sigma) = 2$,

$$D \leq E = S_\Sigma = E_1 \times \cdots \times E_m,$$

where $m = n/2$ and $E_i = \langle e_i \rangle = S_{\Omega - \sigma_i} \cong \boldsymbol{Z}_2$. Further $|E : D| \leq 2$, H_* acts on Σ with orbits $\{\Sigma_1, \ldots, \Sigma_r\}$ of length $s \geq 5$, and $C_E(H_*) = \langle f_1, \ldots, f_r \rangle$, where

$$f_j = \prod_{i \in \Sigma_j} e_i$$

is of order 2. Thus $m_2(C_E(H_*)) = r = m/s \leq m/5 = n/10$. Then (4) completes the proof of (5). $\square$

7.18. *Suppose* $(V, \Gamma) \in \mathfrak{G}$ *and* $V \in \mathfrak{V}$.

(1) $m_2(V \cap D) \geq (n-2)/4$.

(2) $\Gamma \leq \Sigma$, *so* $D \leq \hat{G}_\Gamma$.

(3) H_* *is nontrivial on* Σ_γ *for each* $\gamma \in \Gamma$.

(4) $k(\Gamma) \geq 10$ *is even.*

Proof. As $V \in \mathfrak{V}$, $m_2(V \cap D) \geq m_2(D)/2$. As $k(\Sigma) = 2$, $m_2(D) \geq (n-2)/2$, so (1) holds. By (1), $m_2(V \cap D) > n/10$, so (2) follows from 7.17.5.

By 7.16, $\Gamma \in \mathscr{P}_1$, so $H_* \leq G_\Gamma$ and then as H is transitive on Γ, H_* is nontrivial on γ. As H acts on Σ and $\Gamma \leq \Sigma$, H_* acts on Σ_γ and then as $k(\Sigma) = 2$ and H^* is perfect, (3) holds with $|\Sigma_\gamma| \geq 5$ and then (4) also follows. $\square$

7.19. $\mathfrak{V} \neq \varnothing$.

Proof. Adopt the notation of 7.8. As $W \in \mathscr{C}^!$, $\mathscr{W}_0^* \cap W = \{U_1, \ldots, U_s\}$ is of order $s \geq 2$, so

$$D^* = X_1^* \oplus \cdots \oplus X_s^*$$

by 7.8, where $X_i := U_i \cap D$. Order the U_i so that $m_2(U_i) \geq m_2(U_{i+1})$ for each i, and take $V \in \mathscr{C}^! - \{W\}$ such that $V \cap W = U_1 \cdots U_{s-1}$. Thus

$$m_2(V \cap D) \geq m_2(D)/2. \qquad \square$$

7.20. *Assume* $V \in \mathfrak{V}$ *and* $Y := VH$ *is imprimitive on* Ω. *Let* Γ *be a maximal member of* $\mathscr{P}'(Y)$, $\gamma \in \Gamma$, $r := k(\Gamma)$, *and* $X := V \cap D$. *Then:*

(1) Y^γ *is primitive but not almost simple.*

(2) $r \equiv 0 \mod 4$.

(3) *$z \in J$.*

(4) *$X \leq \hat{G}_\Gamma$ and $m_2(X^\gamma) > (r-2)/4$.*

(5) *If $r = 16$ and $m_2(X^\gamma) = 4$, then X is the direct product of the groups $X_{\Omega-\gamma}$, for $\gamma \in \Gamma$, with $X_{\Omega-\gamma} \cong E_{16}$.*

Proof. By 7.16, we have $\Gamma \in \mathcal{P}_1$. By the maximality of Γ, Y^γ is primitive. By 7.18.2, $X \leq V_\Gamma$, and by 7.18.1, $m_2(X) \geq (n-2)/4$, so $X \neq 1$. Hence Y^γ is not almost simple by 6.5.4, completing the proof of (1).

By 7.18.4, r is even. Then (2) follows from (1) and 1.4. By (2), $z \in A$, so $z \in D$. Further H centralizes Z, so (3) follows from 7.8.

As $X \leq V_\Gamma$ is H-invariant,

$$m_2(X) \leq (n/r) \cdot m_2(X^\gamma),$$

so as $m_2(X) \geq (n-2)/4$, $m_2(X^\gamma) \geq (n-2)r/4n > (r-2)/4$, establishing (4). Finally if $r = 16$ and $m_2(X^\gamma) \leq 4$, then $m_2(X) \leq n/4$, while as $n \equiv 0 \mod 4$, $n/4$ is the smallest integer as big as $(n-2)/4$. We conclude that $m_2(X) = n/4$ and then that (5) holds. □

7.21. *For each $V \in \mathfrak{V}$, VH is primitive on Ω.*

Proof. Assume otherwise and choose notation as in 7.20. We observe that Y^γ, γ, z^γ, X^γ satisfy the hypotheses of 1.9 in the role of G, Ω, z, X by 7.20. Then we conclude from 1.9 that $r = 16$, Y^γ is affine and $X^\gamma \cong E_{16}$. Then by 7.20.5, X is a direct product of the groups $X_{\Omega-\gamma}$. Hence by 1.9, $Y_\Gamma = \langle X^Y \rangle$, so Y is the direct product of the groups $Y_{\Omega-\gamma} \cong Y^\gamma$. As $X \leq V \trianglelefteq Y$, $Y_\gamma \leq V$. But then as $H_*^\gamma \leq Y^\gamma$ for each $\gamma \in \Gamma$, $H_* \leq V$, a contradiction. □

7.22. *Let $V \in \mathfrak{V}$ and $Y := VH$. Then:*

(1) *Y is primitive but not almost simple on Ω.*

(2) *$n \equiv 0 \mod 4$.*

(3) *$z \in J$.*

Proof. By 7.21, Y is primitive on Ω, so (1) follows from 2.14. As $k(\Sigma) = 2$, n is even, so as Y is primitive but not almost simple, (2) follows from 1.4. Now (3) follows as in the proof of 7.20.3. □

We are now in a position to obtain a contradiction to Hypothesis 7.1. We argue as in the proof of 7.21. By 7.19 we can pick $V \in \mathfrak{V}$; set $Y := VH$ and $X := V \cap D$.

By 7.18.1, $m_2(X) \geq (n-2)/4$. Then by 7.22, the hypotheses of 1.9 are satisfied by Y in the role of G, so that lemma says that $n = 16$ and $Y = \langle X^Y \rangle$. Then as $X \leq V \trianglelefteq Y$, $H_* \leq Y = V$, for our final contradiction.

This contradiction shows:

Theorem 7.23. *Assume Hypothesis* 6.1. *Then* $\mathcal{P}_2(H) = \emptyset$, *so* $\mathcal{P}'(H) = \mathcal{P}_1(H)$.

8 The case $\mathcal{P}'(H) = \mathcal{P}_1$

In this section we continue to assume Hypothesis 6.1, and adopt the notation established there.

8.1. *The following hold:*

(1) $\mathcal{P}'(H) = \mathcal{P}_1(H)$.

(2) *Let* Γ *be the set of orbits of* H_* *on* Ω. *Then* Γ *is the greatest member of* $\mathcal{P}'(H)$.

(3) *For each* $\gamma \in \Gamma$, H^γ *is primitive.*

(4) *If* $Y \in \mathcal{O}_G(H)$ *is imprimitive, then there is a greatest member* Γ_Y *of* $\mathcal{P}'(Y)$, *and* $\Gamma_Y \leq \Gamma$.

Proof. Part (1) is a restatement of Theorem 7.23. Then (2) follows from parts (1) and (2) of 6.5. As Γ is maximal in $\mathcal{P}'(H)$, (3) and (4) follow. □

8.2. *Assume that for* $\gamma \in \Gamma$ *that* $J_{\Omega-\gamma} \neq 1$. *Then for each* $W \in \mathcal{W}_0'$,

(1) WH *is imprimitive on* Ω.

(2) $\mathcal{P}'(WH)$ *has a greatest member* Γ_W, *and* $\Gamma_W \leq \Gamma$.

Proof. Assume $I := J_{\Omega-\gamma} \neq 1$. Then as $I^\gamma \trianglelefteq H^\gamma$, it follows that I^γ is transitive by 8.1.3. Let $W \in \mathcal{W}_0$ and $Y := WH$. Suppose Y is primitive on Ω. Then by 3.5 in [8] and 2.14, Y is the stabilizer of an affine structure on Ω, $J \not\leq F^*(Y)$, and $Y = \langle F^*(Y), J^Y \rangle$. As $F^*(Y)$ is the unique minimal normal subgroup of Y and $1 \neq W \trianglelefteq Y$, $F^*(Y) \leq W$. Then as $J \leq W$, $Y = \langle F^*(Y), J^Y \rangle \leq W$. But now $H_* \leq Y \leq W$, a contradiction. Thus (1) holds. Then (1) and 8.1.4 imply (2) with $\Gamma_W = \Gamma_Y$. □

8.3. *For each* $\gamma \in \Gamma$, $J_{\Omega-\gamma} = 1$.

Proof. Assume otherwise. By 3.3, there exist $W \in \mathcal{W}_0'$ with $Y := WH \not\leq M(\Gamma)$. Set $\Sigma := \Gamma_W$; by 8.2, $\Sigma < \Gamma$. Let $\gamma \subseteq \sigma \in \Sigma$, $I := J_{\Omega-\gamma}$ and $X := \langle I^{N_Y(\sigma)} \rangle$. Then $X \leq W$. By maximality of Σ in $\mathcal{P}'(Y)$, Y^σ is primitive. Hence by 3.5 in

[8] and 6.5.4, Y^σ is the stabilizer of an affine structure on σ, $I^\sigma \not\leq F^*(Y^\sigma)$, and $Y^\sigma = \langle F^*(Y^\sigma), I^{\sigma Y^\sigma}\rangle$. As $F^*(Y^\sigma)$ is the unique minimal normal subgroup of Y^σ and $1 \neq X^\sigma \trianglelefteq Y^\sigma$, $F^*(Y^\sigma) \leq X^\sigma$. Then as $I \leq X$,

$$Y^\sigma = \langle F^*(Y^\sigma), I^{\sigma Y^\sigma}\rangle \leq X^\sigma.$$

Now as $X \leq W_{\Omega-\sigma}$ and $H_*^\sigma \leq Y^\sigma$, $H_* \leq \langle X^H\rangle \leq W$, a contradiction. □

In the remainder of this section (until the last theorem) we assume the following hypothesis:

Hypothesis 8.4. $J_E := J \cap D(\Gamma) \neq 1$.

In addition adopt the following notation: Set $M := M(\Gamma)$, $E := D(\Gamma)$, and for $\gamma \in \Gamma$, let $E(\gamma) := A_{\Omega-\gamma}$ (a component of E), $\pi_\gamma : E \to E(\gamma)$ the projection map, $X_\gamma \leq J_E\pi_\gamma$ such that X_γ^γ is a minimal normal subgroup of H^γ and H permutes $\{X_\gamma : \gamma \in \Gamma\}$ via conjugation, and $X := \langle X_\gamma : \gamma \in \Gamma\rangle$. Set $U := \bar{X}$ and $k = k(\Gamma) := |\gamma|$.

8.5. *Let $\gamma \in \Gamma$. Then:*

(1) *H^γ is primitive and $X_\gamma \cong X_\gamma^\gamma$ is transitive on γ.*

(2) *$U \in \mathcal{W}_0^!$.*

(3) *Let $\mathcal{W}_0^* \cap U = \{U(i) : 1 \leq i \leq s\}$ and for $\alpha \subseteq \{1,\dots,s\}$, set*

$$U(\alpha) := \langle U(i) : i \in \alpha\rangle \quad \textit{and} \quad X(\alpha) := U(\alpha) \cap X.$$

Then $U(\alpha) = \bar{X}(\alpha)$, $X(\alpha) = \langle X(i) : i \in \alpha\rangle$, and $\bigcap_i X(i) = J \cap X =: J_X$.

(4) *For $\gamma \in \Gamma$ and $1 \leq i \leq s$, $J_X\pi_\gamma = X(i)\pi_\gamma = X_\gamma$.*

(5) *$\mathcal{W}_0(\leq U) = \Xi_M$.*

Proof. Observe that Hypothesis 4.1 is satisfied by $M := M(\Gamma)$ and

$$\mathcal{L} := \{E(\gamma) : \gamma \in \Gamma\}.$$

Further in the language of Notation 4.2, in this setup, $X = \tilde{J}_X$, where J_X is the preimage in J_E under π_γ of a minimal normal subgroup of H^γ. Moreover we have $J_X = J \cap X$. By 4.4.2, $\tilde{J}J \in \mathcal{W}_0$, so as $X \leq \tilde{J}$ is H-invariant, $U = \bar{X} \in \mathcal{W}_0$ by 2.7. By 8.2.3, H^γ is primitive, so as $1 \neq X_\gamma^\gamma \trianglelefteq H^\gamma$, X_γ is transitive on γ. Thus (1) holds, and if $U \leq W \in \mathcal{W}_0$, then arguing as in the proof of 8.2, $Y := WH$ is imprimitive on Ω, there is a greatest member Γ_W of $\mathcal{P}'(Y)$, and $\Gamma_W \leq \Gamma$. Then arguing as in the proof of 8.3, $\Gamma_W = \Gamma$.

Suppose (2) fails, and let $\{W_i : 1 \leq i \leq s\} = \{W \in \mathcal{W}^! : U \leq W\}$. By the previous paragraph, $W_i \leq M$. Thus the connected component $\mathcal{C}$ of U is contained in Ξ_M, so by 4.6.4, $X \leq J$, contrary to 8.3.

Therefore (2) holds. Thus as $U = \bar{X}$, we conclude from 3.7 that (3) holds. By construction of X, $J_X\pi_\gamma = X_\gamma = X\pi_\gamma$, so as $J_X = J \cap X \leq X(i) \leq X$, part (4) follows from (3).

Recall $X = \tilde{J}_X$ and by 8.3, $X \not\leq J$. Thus from 4.6.3, Ξ_M is connected, but does not contain the connected component of U. Thus (5) follows from (2). □

8.6. *Let $\gamma \in \Gamma$. Then either:*

(1) *$k = p^a$ is a prime power, H^γ is affine, and $X_\gamma \cong X^\gamma = F^*(H^\gamma) \cong E_{p^a}$ is regular on γ. Further $X = X(1) \cdots X(s)$, and for each $1 \leq i \leq s$, $X(i)$ is normal in UH.*

(2) *X^γ is the direct product of components transitively permuted by H^γ isomorphic to some simple group X_0, and X is the direct product of a set $\mathcal{X}$ of components transitively permutated by H and isomorphic to X_0. For $\alpha \subseteq \{1, \ldots, s\}$ there exist an H-invariant partition $\Theta(\alpha)$ of $\mathcal{X}$ such that $X(\alpha)$ is the direct product of full diagonal subgroups of X_θ, $\theta \in \Theta(\alpha)$, and $X_\theta = \prod_{Y\in\theta} Y$. Thus $X(\alpha)$ is the direct product of its components, each of which is isomorphic to X_0, and H is transitive on these components.*

Proof. As $X_\gamma \cong X_\gamma^\gamma$ is a minimal normal subgroup of the primitive group H^γ, either

(i) $k = p^a$ is a prime power, H^γ is affine, and $X_\gamma^\gamma = F^*(H^\gamma) \cong E_{p^e}$ is regular on γ, or

(ii) X_γ^γ is the direct product of components permuted transitively by H^γ and isomorphic to some nonabelian simple group X_0.

Then as H is transitive on Γ and X is the direct product of the groups X_γ, $\gamma \in \Gamma$, it follows that X is an elementary abelian p-group in (i), while in (ii), X is the direct product of its set $\mathcal{X}$ of components, with H transitive on $\mathcal{X}$ and the members of $\mathcal{X}$ are isomorphic to X_0. Then in (i), (1) holds by 8.5.3. Similarly in (ii), (2) holds using 8.5.4 and 1.4 in [7]. □

Notation 8.7. Assume Hypothesis 8.4, adopt the notation of 8.5.3, and let $\mathcal{C}$ be the connected component of Λ' containing U. By 8.5.2, there is a unique $V \in \mathcal{W}_0^* \cap \mathcal{C}$ with $V \not\leq U$. Let

$$\mathcal{W}^! \cap \mathcal{C} =: \mathcal{C}^! \quad \text{and} \quad \mathcal{V} := \{W \in \mathcal{C} : W \not\leq U\}.$$

Thus $\mathcal{C}^! - \{U\} \subseteq \mathcal{V}$, and for each $W \in \mathcal{V}$, $V \leq W$.

For $W \in \mathcal{V}$, set $Y_W = WH$. If Y_W is primitive on Ω, let $\Gamma_W = 0$ and $\Phi_W = \Omega$. If Y_W is not primitive on Ω, then by 8.1.4 there is a greatest member $\Gamma_W = \Gamma_{Y_W}$ of $\mathcal{P}'(Y_W)$, and we have $\Gamma_W \leq \Gamma$. In this case let $\Phi_W \in \Gamma_W$. Set $B_W = F^*(Y_W)$ if $\Phi_W = \Omega$, and let B_W be the product of the H-conjugates of the inverse image in $A_{\Omega-\Phi_W}$ of $F^*(Y_W)^{\Phi_W}$ if $\Phi_W \neq \Omega$.

8.8. *Assume Hypothesis* 8.4 *and adopt Notation* 8.7. *Let* $W \in \mathcal{V}$. *Then:*

(1) $Y_W^{\Phi_W}$ *is primitive but not almost simple.*

(2) Γ_{Φ_W} *is a* $(UH)^{\Phi_W}$ *invariant partition of* Φ_W.

(3) *Suppose* $\Phi_W \neq \Omega$. *Let* $\alpha \in \Gamma_W$, *and* $\pi_\alpha : D(\Gamma_W) \to A_{\Omega-\alpha}$ *be the projection. Then* $D(\Gamma) \leq D(\Gamma_W)$, *so* $\langle U, W\rangle \leq M(\Gamma_W)$. *Further for each* $P \in \Xi_{M(\Gamma_W)}$, $(P \cap D(\Gamma_W))\pi_\alpha \leq P$.

(4) $B_W \leq W$.

(5) $B_W \not\leq U$.

(6) *If* $W \notin \mathcal{W}_0^!$, *then* $B_W \neq B_P$ *for some* $P \in \mathcal{W}_0$ *with* W *maximal in* P.

(7) *Suppose* $V \leq Q \leq W$ *with* $Q \in \mathcal{W}_0$. *Then* $\Gamma_Q = \Gamma_W$, $Y_Q^{\Phi_W}$ *is primitive, and* $B_Q \leq B_W$.

(8) *There exists* $W \in \mathcal{C}^!$ *and* $V = V_1 < \cdots < V_s = W$ *with* $V_i \in \mathcal{W}_0$ *and* $\bar{B}_{V_i} = V_i$ *for each* $1 \leq i \leq s$.

Proof. Let $\Phi := \Phi_W$, $Y := Y_W$, $B := B_W$. If $\Phi = \Omega$, then, by definition of Φ, Y is primitive on Ω, so Y^Φ is primitive, while Y^Φ is not almost simple by 2.14. If $\Phi \neq \Omega$, then Y^Φ is primitive as $\Phi \in \Gamma_W$ and Γ_W is the greatest member of $\mathcal{P}'(Y)$. Further Y^Φ is not almost simple by 8.1.1 and 6.5.4. Thus (1) is established.

As $\Gamma \in \mathcal{P}'(H)$, and $\Gamma_W \leq \Gamma$ when $\Phi \neq \Omega$, (2) follows.

Assume the setup of (3). By 8.1.4, $\Gamma_W \leq \Gamma$, so $D(\Gamma) \leq D(\Gamma_W)$. Therefore $X \leq D(\Gamma) \leq M(\Gamma_W) =: M_W$, so $U = JX \leq M_W$, and hence $\langle U, W\rangle \leq M_W$ as $\Gamma_W \in \mathcal{P}'(HW)$. Observe that Hypothesis 4.1 is satisfied by M_W, $D(\Gamma_W)$ in the roles of "M, D". Further as $\langle U, W\rangle \leq M_W$, each group in the connected component of U is contained in M_W. Therefore (3) follows from 4.6.4.

By definition in 8.7, $B^\Phi = F^*(Y^\Phi)$. Hence by (1), either B^Φ is the unique minimal normal subgroup of Y^Φ, or Y^Φ is doubled. As $1 \neq W^\Phi \trianglelefteq Y^\Phi$, we conclude that either $B^\Phi \leq W^\Phi$ or Y^Φ is doubled, with $B^\Phi = F_1^\Phi \times F_2^\Phi$ the product of the two minimal normal subgroups F_j^Φ of Y^Φ, and $F_1^\Phi = B^\Phi \cap W^\Phi$. Now in the second case, as

$$H_*W/W = F^*(Y/W) \cong L,$$

it follows that $W^\Phi H_*^\Phi = W^\Phi F_2^\Phi$, and then $F_1^\Phi \cong F_2^\Phi \cong L$. Also

$$[F_2^\Phi, W^\Phi] \le F_2^\Phi \cap W^\Phi = 1,$$

so F_2^Φ centralizes W^Φ. Therefore if $H_*^\Phi = F_2^\Phi$ then H_* centralizes W. But then by 6.5.2, $W \le M$, contrary to 8.5.5. Thus $H_*^\Phi \ne F_2^\Phi$, so $\bar{F}_2 \in \mathcal{W}_0$ by 2.12, where F_2 is the product of the H-conjugates of the inverse image of F_2^Φ in $A_{\Omega-\Phi}$ in case $\Phi \ne \Omega$. Further $[F_2, W] = 1$, so as $J_X \le W$ and as Γ is the set of orbits of J_X on Ω, we have $F_2 \le M$. Thus $\bar{F}_2 \le U$ by 8.5.5. We saw that H is transitive on the components of F_2, so F_2 is a minimal normal subgroup of $F_2 H$. Therefore, in the notation of 8.5.3, $F_2 \le X(i)$ for some $1 \le i \le s$, and then from 8.5.1 and 8.5.4, Γ is the set of orbits of F_2 on Ω. As W centralizes F_2, it follows that $W \le M$, contradicting 8.5.5.

Thus $B^\Phi \le W^\Phi$. Then (4) is immediate from the definition of B if $\Phi = \Omega$, while if $\Phi \ne \Omega$, it follows from (3).

Assume $B \le U$. Claim $T := X \cap B \ne 1$. For if not, $[X \cap W, B] \le X \cap B = 1$, so $X \cap W \le C_W(B) \le B$ as $B^\Phi = F^*(Y^\Phi)$. In particular

$$1 \ne J_X \le X \cap W \le T,$$

establishing the claim.

Now, using (4), $B \le U \cap W \le N_W(X) \le N_W(T)$. Also T is H-invariant, so $\bar{T} \in \mathcal{W}_0$ by 2.7, and as $T \ne 1$, T^γ is transitive by 8.5.1, so Γ is the set of orbits of T on Ω. Adopt the notation of 3.7 with B, Y in the roles of Y, X. Then we have $\mathcal{D} = \mathcal{W}_0(\le W)$ and $\bar{T} = L_\alpha$ for some $\alpha \subseteq \mathcal{A}$, so by 3.7, $L_{\mathcal{B}}$ acts on $T = L_\alpha \cap B$, so T is normal in $\langle B, L_{\mathcal{B}} \rangle = \langle L_{\mathcal{A}}, L_{\mathcal{B}} \rangle = L_{\mathcal{D}} = W$. Therefore W acts on the set Γ of orbits of T, so $W \le M$, contrary to 8.5.5. This completes the proof of (5).

Assume (6) fails and let $\mathcal{D} := \mathcal{C} \cap \mathcal{W}_0^*$. From 3.4, $W = L_\alpha$ for some $\alpha \subseteq \mathcal{D}$, and the members of $\mathcal{W}_0$ in which W is maximal are the groups $L(b) = L_{\alpha \cup \{b\}}$, $b \in \mathcal{B} = \mathcal{D} - \alpha$. By hypothesis, $B = B_{L(b)}$ for each $b \in \mathcal{B}$, so $L_{\mathcal{D}}$ acts on B, contrary to 3.8. This establishes (6).

Assume the hypothesis of (7). By (3), U acts on Γ_Q, so $W \le \langle U, Q \rangle$ acts on Γ_Q, and then by definition of Γ_W, $\Gamma_W = \Gamma_Q$. Therefore also $\Phi_W = \Phi_Q$, so $Y_Q^{\Phi_W}$ is primitive by (1). By (5), $\bar{B}$ and $\bar{B}_V$ are not contained in U, while $\bar{B}_V \le V$ by (4), so as $V \cap U = J$ and $V \in \mathcal{W}_0^*$, $\bar{B}_V = V$, and by 3.7, $V \cap B \not\le J$. Thus $Q \cap B \ne 1$, so

$$1 \ne Q^\Phi \cap B^\Phi \trianglelefteq Y_Q^\Phi.$$

Then as $B_Q^\Phi = F^*(Y_Q^\Phi)$ and $Y_Q \le Y$,

$$1 \ne B_Q^\Phi \cap B^\Phi.$$

Then by 4.3 in [4], $B_Q^\Phi \le B^\Phi$, so $B_Q \le B$, completing the proof of (7).

We saw $\bar{B}_V = V$. Let $V = V_1 < \cdots < V_r = W$ be a chain of maximal length subject to $\bar{B}_{V_i} = V_i$, and suppose $W \notin \mathcal{W}_0^!$. Then for $P \in \mathcal{W}_0$ with W maximal in P, $\bar{B}_P \neq P$. However by (7), $B \leq B_P$, so $W = \bar{B} \leq \bar{B}_P$, and hence $W = \bar{B}_P$ by maximality of W in P. This contradicts (6), and completes the proof of (8). □

8.9. *Choose W as in* 8.8.8 *and set $Y := Y_W$, $\Phi := \Phi_W$, and $B := B_W$. Then:*

(1) $B = \langle B \cap U, B_V \rangle$ *and*

$$(B \cap U) \cap B_V = B \cap J$$

with $\overline{B \cap U}$ maximal in $\mathcal{W}_0(< U)$ and $\mathcal{W}_0(< W)$, $\bar{B}_V = V$, and $\bar{B} = W$.

(2) $Y = BH$.

(3) *Y^Φ is not affine.*

(4) *Y_V^Φ is primitive.*

(5) *Y^Φ is semisimple, and hence Y^Φ preserves a regular product structure $\mathcal{F}$ on Φ. Further B^Φ is the direct product of the set $\mathcal{L}$ of components of Y^Φ, each of which is simple, and $\mathcal{L}$ is permuted transitively by H^Φ.*

(6) *H_* acts nontrivially on $\mathcal{L}$.*

Proof. First 8.8.8 and 3.7 imply (1). By (1), $W = \bar{B} = BJ$, so

$$Y = WH = BJH = BH,$$

establishing (2).

Suppose Y^Φ is affine. Then B^Φ is an elementary abelian p-group for some prime p, and Y^Φ is irreducible on B^Φ. But by (2), $Y = BH$, so H^Φ is irreducible on B^Φ. However by (1), B_V is a proper nontrivial subgroup of B, while by 8.8.7, $\Gamma_V = \Gamma_W$, so B_V^Φ is a proper nontrivial subgroup of B^Φ. However B_V^Φ is H^Φ-invariant, contradicting H^Φ irreducible on B^Φ. Hence (3) holds.

Part (4) follows from 8.8.7.

By (1), $\bar{B}_V = V$ and $\bar{B} = W$, so as $V \neq W$ also $B_V \neq B$. From the proof of 8.8.7, $B_V^\Phi \leq B^\Phi$, so as $B_V \neq B$ we have $B_V^\Phi < B^\phi$.

By (3) and as Y^Φ is primitive, B^Φ is the direct product of a set $\mathcal{L}$ of simple components. Further H^Φ is transitive on $\mathcal{L}$ iff Y^Φ is not doubled.

Suppose Y^Φ is doubled. Then $B^\Phi = B_1^\Phi \times B_2^\Phi$ is the product of the two minimal normal subgroups of Y^Φ. By (4) and Proposition 8 in [4], and as

$$F^*(Y_V^\Phi) = B_V^\Phi \neq B^\Phi,$$

it follows that we may choose $B_V^\Phi = B_1^\Phi$. As $C_{Y_V^\Phi}(B_V) = 1$,

$$Y_V^\Phi \cap B^\Phi = B_1^\Phi.$$

Now B_2^Φ is an H^Φ-invariant subgroup of B^Φ, so by 3.7, $B_2^\Phi \leq U^\Phi$. But then $(UH)^\Phi$ is primitive, contradicting 8.8.2. Therefore H^Φ is transitive on $\mathcal{L}$ and Y^Φ is not doubled.

As $B_V^\Phi < B^\Phi$, Y^Φ is not complemented by Proposition 10 in [4]. By Proposition 6 in [4] and as H^Φ is transitive on $\mathcal{L}$, Y^Φ is not diagonal. Hence (5) holds.

By 8.1.2, H_* is trivial on Γ, so as $\Gamma_W \leq \Gamma$, $H_* \leq Y_{\Gamma_W}$. By (1), $\bar{B} = W$, so $H_* \not\leq B$. Let K be the kernel of the action of Y on its components. Then K/B is solvable, so as $H_* \not\leq B$, also $H_* \not\leq K$. Hence (6) follows from the transitivity of H on the components of Y. □

8.10. *Choose W as in* 8.8.8*, set $\Phi := \Phi_W$, and let*

$$\mathcal{R}(\Phi) := \{g \in \mathrm{Sym}(\Phi) : |\mathrm{Fix}_\Phi(g)| \geq |\Phi|/2\}.$$

Then:

(1) *$U \cap W = U(\alpha)$ and $X \cap W = X(\alpha)$, for some maximal proper subset α of $\{1, \ldots, s\}$.*

(2) *Assume* 8.6.2 *holds. Then $X(\alpha) = \langle X(\alpha) \cap \mathcal{R}(\Phi)\rangle$.*

Proof. Part (1) is a consequence of 8.5.3 and 3.4.

Assume 8.6.2 holds, adopt the notation of 8.6.2, and set

$$\mathcal{X}^\Phi := \{F^\Phi : F \in \mathcal{X} \text{ and } F^\Phi \neq 1\}.$$

For $\gamma \in \Gamma_\Phi$, set $\hat{\gamma} := \{F^\Phi \in \mathcal{X}^\Phi : F^\Phi \leq E(\gamma)^\Phi\}$. Then $\hat{\Gamma} = \{\hat{\gamma} : \gamma \in \Gamma_\Phi\}$ is a partition of $\mathcal{X}^\Phi$ with

$$|\hat{\Gamma}| = |\Gamma_\Phi| =: m.$$

By 8.6, for each $\beta \subseteq \alpha$, $X(\beta)^\Phi$ is the direct product of components F_θ^Φ, where $\theta \in \Theta_{\beta,\Phi}$, such that $\Theta_{\beta,\Phi}$ is an H^Φ-invariant partition of $\mathcal{X}^\Phi$, and F_θ^Φ is a full diagonal subgroup of X_θ^Φ. As F_θ^Φ is simple and $\pi_\gamma : X(\beta)^\Phi \to X_\gamma^\Phi$ is a surjection, $F_\theta^\Phi \pi_\gamma$ is a component of X_γ^Φ whenever $F_\theta^\Phi \pi_\gamma \neq 1$. Thus $\hat{\Gamma} \vee \Theta_{\beta,\Phi} = \infty$, and hence $k_\beta = |\theta| \leq m$. Now for $\delta \subseteq \beta$, $X(\delta) \leq X(\beta)$, so $\Theta_{\beta,\Phi} \leq \Theta_{\delta,\Phi}$. It follows that $m \geq k_\varnothing \geq 2^{s-1} k_\alpha$, so $k_\alpha \leq m/2^{2-1} \leq m/2$.

Take $\beta = \alpha$. Now $F_\theta^\Phi \leq A_{\Phi-\mu}$, where

$$\mu = \bigcup_{\{\gamma : F_\theta^\Phi \pi_\gamma \neq 1\}} \gamma,$$

and μ is of order $kk_\alpha \leq km/2 = |\Phi|/2$, so $F_\theta^\# \subseteq \mathcal{R}(\Phi)$, completing the proof of the lemma. □

8.11. *Assume* 8.6.1 *holds and choose* W *as in* 8.8.8*, set* $Y := Y_W$*,* $\Phi := \Phi_W$*, and* $B := B_W$*, and let* K *be the kernel of the action of* Y^{Φ} *on the product structure* $\mathcal{F}$ *of* 8.9.5*. Then:*

(1) $(X \cap W)^{\Phi} \leq K$.

(2) $X_B := (X \cap B)^{\Phi} \neq 1$.

(3) X_B *contains its projection on each member of* $\mathcal{L}$.

Proof. By 8.10.1, $X \cap W = X(\alpha)$ for some maximal proper subset α of $\{1, \dots, s\}$, and by 8.5.3, $U \cap W = U(\alpha) = \bar{X}(\alpha)$. Therefore $X \cap W \neq 1$.

Let $\gamma \in \Gamma$, $I := (X \cap W)^{\Phi}$, and suppose P is a nontrivial H-invariant subgroup of $C_{M^{\Phi}}(I)$. Then $1 \neq I^{\gamma} \trianglelefteq H^{\gamma}$, so by 8.5.4, $I^{\gamma} = X^{\gamma}$. Thus

$$P^{\gamma} \leq C_{M^{\gamma}}(X^{\gamma}) = X^{\gamma},$$

so $P_{\Gamma_{\Phi}} \leq C_{M^{\Phi}}(X^{\Phi}) = X^{\Phi}$, and $N_P(\gamma) \leq C_{M^{\Phi}}(X_{\gamma})$, so that

$$X_P := C_{X^{\Phi}}(P) = \prod_{\sigma \in \Sigma} X_{\sigma},$$

where Σ is the set of orbits for P on Γ_{Φ} and X_{σ} is a full diagonal subgroup of $\langle X_{\gamma} : \gamma \in \sigma \rangle$.

Suppose $P \not\leq X^{\Phi}$. Then $P \neq P_{\Gamma_{\Phi}}$, so $\Sigma \neq \Gamma_{\Phi}$, and hence $X_P \neq X^{\Phi}$. Hence $J_X^{\Phi} \leq (X \cap W)^{\Phi} \leq X_P < X^{\Phi}$, and X_P is H^{Φ}-invariant, so

$$X_P = (X \cap W)^{\Phi} = I$$

by 3.7 and maximality of $(X \cap W)^{\Phi} = X(\alpha)^{\Phi}$ in X^{Φ}. Further either

(i) P is transitive on Γ_{Φ}, or

(ii) $(X_{\sigma}^{\Phi})^{\#} \subseteq \mathcal{R}(\Phi)$, so $I = \langle X_P \cap \mathcal{R}(\Phi) \rangle \leq K$,

by 8.9.5 and 4.2 in [8]. Indeed if (i) holds, then $X_P \cong X_{\gamma}$ as a $N_H(\gamma)$-module, so $N_H(\gamma)$ is irreducible on X_P, and hence $X_P = J_X^{\Phi}$, contradicting

$$J_X^{\Phi} < X(\alpha)^{\Phi} = X_P.$$

Thus (ii) holds if $P \not\leq X^{\Phi}$.

Claim $X_B := I \cap B^{\Phi} \neq 1$. Assume otherwise. Then $[I, (B \cap U)^{\Phi}] \leq X_B = 1$. Also $(B \cap U)^{\Phi} \neq 1$ by 8.9.1, so $(B \cap U)^{\Phi} \not\leq X^{\Phi}$ or else $(B \cap U)^{\Phi} \leq I \cap B^{\Phi} = 1$. Thus applying remarks above to $(B \cap U)^{\Phi}$ in the role of P, we conclude (ii) holds, so $I \leq K$. Then as I centralizes $(B \cap U)^{\Phi}$, it also centralizes the product $\tilde{B}$ of the projections of $(B \cap U)^{\Phi}$ on the various members of $\mathcal{L}$. Hence applying remarks above to $\tilde{B}$ in the role of P, $\tilde{B}_{\Gamma_{\Phi}} \leq X^{\Phi}$, so that $\tilde{B}_{\Gamma_{\Phi}} \leq X^{\Phi} \cap B^{\Phi} = X_B = 1$.

Then as $[H_*, \tilde{B}] \leq \tilde{B}_{\Gamma_\Phi}$, H_* centralizes $\tilde{B}$, which is not the case as H_* acts nontrivially on $\mathcal{L}$ by 8.9.6. Thus the claim is established, so (2) holds.

Next let $\tilde{I}$ be the product of the projections of X_B on the various members of $\mathcal{L}$. Now X^Φ is abelian, so X_B and hence also $\tilde{I}$ are abelian. Also $1 \neq X_B$ is H^Φ-invariant, so by 8.5.1, X_B^γ is transitive for each $\gamma \in \Gamma_\Phi$, and hence Γ_Φ is the set of orbits of X_B on Φ, and X_B^γ is regular. Thus as I centralizes X_B^γ, $X_B^\gamma = I^\gamma$, so I is contained in the product X_0 of the projections of X_B on the X_γ, and $X_0 \leq X$. Similarly $\tilde{I}$ acts on Γ_Φ and

$$\tilde{I}_{\Gamma_\Phi} \leq X_0 \cap B^\Phi = X_B.$$

Thus $[I, \tilde{I}] \leq \tilde{I}_{\Gamma_\Phi} \leq X_B$, so we have quadratic action of I on $\tilde{I}$. It follows that either (1) holds, or $p = 2$ and all orbits of I on $\mathcal{L}$ are of length 2, and we may assume the latter. Therefore $|\tilde{I} : X_B| = |X_B|$. But $[H_*, \tilde{I}] \leq \tilde{I}_{\Gamma_\Phi} \leq X_B$, so as H_* is nontrivial on $\mathcal{L}$ by 8.9.6, $|\tilde{I} : X_B| < |X^B|$, a contradiction. This completes the proof of (1).

Finally suppose (3) fails, so that $\tilde{I} \neq X_B$, and hence $\tilde{I} \not\leq X^\Phi$. From the previous paragraph, $\tilde{I}_{\Gamma_\Phi} \leq X_B$, so $\tilde{I} \neq \tilde{I}_{\Gamma_\Phi}$. On the other hand X^Φ acts on X_B and hence on $\tilde{I}$, so $[X^\Phi, \tilde{I}] \leq \tilde{I}_{\Gamma_\Phi} \leq X_B$. Therefore X^Φ acts quadratically on $\tilde{I}$, and we obtain a contradiction as in the previous paragraph. □

8.12. *Assume* 8.6.2 *holds, choose W as in* 8.8.8 *and set* $\Phi := \Phi_W$ *and* $B := B_W$. *Then:*

(1) $(X \cap W)^\Phi \leq B^\Phi$.

(2) $(X \cap W)^\Phi$ *contains its projection on each member of* $\mathcal{L}$.

Proof. Part (1) follows from 8.10.2, 8.9.5, and 4.2 in [8].

By 8.10.1, $X \cap W = X(\alpha)$ for a subset α of $\{1, \ldots, s\}$, so by 8.6.2, $X \cap W = E(X \cap W)$. Let $\tilde{X}$ be the product of the projections of $(X \cap W)^\Phi$ on the various members of $\mathcal{L}$. Claim $\tilde{X} \neq B^\Phi$. Suppose otherwise. Then for P a component of $(X \cap W)^\Phi$, P projects on a component Q of B^Φ. Now by 8.9.5, we can identify Φ with a product

$$\Phi = \prod_{Q \in \mathcal{L}} \Phi_Q,$$

where the Φ_Q are m-sets. There exists $x \in Q$ acting without fixed points on Φ_Q. Let $y \in P$ project on x. Then y has no fixed points on Φ, contrary to the proof of 8.10, which showed $P^\# \subseteq \mathcal{R}(\Phi)$. This establishes the claim.

By 8.9.1, $B \cap U$ is maximal in $\mathcal{W}_0(< W)$, so by 3.7.3, $H^\Phi(B \cap U)^\Phi$ is maximal in $B^\Phi H^\Phi$. Therefore as $\tilde{X}$ is normalized by $H^\Phi(B \cap U)^\Phi$, it follows from the claim that $\tilde{X} \leq (B \cap U)^\Phi$. Thus $P \trianglelefteq \tilde{X}$, so P is one of the components of $\tilde{X}$, and then $(X \cap W)^\Phi = \tilde{X}$, completing the proof of the lemma. □

Theorem 8.13. $J \cap D(\Gamma) = 1$.

Proof. Assume otherwise. Then Hypothesis 8.4 is satisfied. Adopt Notation 8.7, pick W as in 8.8.8, and set $Y := Y_W$, $\Phi := \Phi_W$, and $B := B_W$. By 8.9.5, Y^Φ preserves a regular (m, r)-product structure on Φ for some $m \geq 5$ and $r \geq 2$, so we can make an identification

$$\Phi = \prod_{j=1}^{r} \Phi_j$$

where Φ_j is an m-set. If 8.6.1 holds, define X_B as in 8.11.2, while if 8.6.2 holds, then let $X_B = (X \cap W)^\Phi$. Then by 8.11 and 8.12, $X_B \leq B^\Phi$, and X_B is the product of its projections on the various members of $\mathcal{L}$.

For $1 \leq j \leq r$, let Γ_j be the set of orbits of X_B on Φ_j. Let $\mathcal{A} = \Gamma_1 \times \cdots \times \Gamma_r$ and for $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{A}$, define $\gamma_\alpha := \alpha_1 \times \cdots \times \alpha_r \subseteq \Phi$. Then as X_B is the product of its projections:

(1) $\{\gamma_\alpha : \alpha \in \mathcal{A}\}$ is the set Γ'_Φ of orbits of X_B on Φ.

However as $1 \neq X_B$ is H-invariant, and for each $\gamma \in \Gamma_\Phi$, H^γ is primitive by 8.1.3, it follows that X_B^γ is transitive, so:

(2) $\Gamma_\Phi = \Gamma'_\Phi$.

Pick $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{A}$. By 8.9.6, H_* acts nontrivially on $\mathcal{L}$. Thus identifying $\mathcal{L}$ with $\{1, \ldots, r\}$, there exists $h \in H_*$ with $1h = j \neq 1$. As $|\Gamma| > 1$, $\alpha_j \neq \Phi_j$, so there exits $\delta_j \in \Gamma_j - \{\alpha_j\}$. Pick a $\beta \in \mathcal{A}$ with $\beta_1 = \alpha_1$ and $\beta_j = \delta_j$. As Γ_Φ is the set of orbits of H_* on Φ, H_* acts on each member of Γ_Φ, so in particular $\gamma_\alpha h = \gamma_\alpha$. Hence $\alpha_1 h = \alpha_j$. Similarly $\beta_1 h = \beta_j$. But by construction, $\alpha_1 = \beta_1$ while $\alpha_j \neq \beta_j$, a contradiction. This completes the proof of the theorem. □

9 The case H_* simple

In this section we continue to assume Hypothesis 6.1, and adopt the notation established there. By 8.1.1, $\mathcal{P}'(H) = \mathcal{P}_1(H)$. Let Γ be the set of orbits of H_* on Ω. Then by 8.1.2, Γ is the greatest member of $\mathcal{P}'(H)$. Set $D := D(\Gamma)$ and let $\mathcal{L}$ the set of components of D. Thus H is transitive on $\mathcal{L}$, the members of $\mathcal{L}$ are simple, and D is the direct product of the members of $\mathcal{L}$.

9.1. *The following hold:*

(1) $J \cap D = 1$.

(2) $H_* \cong L$.

(3) $H_0 = H_* \times J$.

(4) $W_* = C_{\hat{G}}(H_*) \in \mathcal{W}_0$.

Proof. Part (1) is a restatement of Theorem 8.13. Then as $H_* \le D$ by 8.1, it follows that $H_* \cap J = 1$. Thus $H_* \cong H_*J/J = H_0/J \cong L$, so (2) holds. Further $[H_*, J] \le D \cap J = 1$, so (3) follows.

Let $W_* := C_{\hat{G}}(H_*)$. By (3), $J \le W_*$. On the other hand H acts on W_*, and $H \cap W_* \le C_H(H_*) = C_{H_0}(H_*) = J$ by (3), so $W_* \in \mathcal{W}_0$, establishing (4). □

9.2. *The following hold:*

(1) H_* *is not regular on* $\gamma \in \Gamma$.

(2) $W_* = 1$.

(3) $J = 1$.

(4) H *is almost simple with* $F^*(H) = H_*$.

Proof. Suppose H_* is regular on $\gamma \in \Gamma$. Then $C_{\mathrm{Sym}(\gamma)}(H_*^\gamma) \cong H_* \cong L$ is also regular on γ. Let $W(\gamma) := C_{A_{\Omega-\gamma}}(H_*)$ and $W_0 := \langle W(\gamma) : \gamma \in \Gamma \rangle$. Then

$$W(\gamma) \cong W(\gamma)^\gamma = C_{\mathrm{Sym}(\gamma)}(H_*^\gamma) \cong L,$$

and W_0 is the direct product of the $W(\gamma)$. Further $W_0 \trianglelefteq W_*$, and there is a complement T to W_0 in W_* acting faithfully as $\mathrm{Sym}(\Gamma)$ on Γ, with $N_T(\gamma)$ centralizing $W(\gamma)$; that is W_* is the wreath product of L by $\mathrm{Sym}(\Gamma)$.

By 8.1.3, H^γ is primitive. Thus as $H_*^\gamma \cong L$ is a minimal normal subgroup of H^γ, either H^γ is almost simple, or H^γ is doubled with minimal normal subgroups H_*^γ and $W(\gamma)^\gamma$. In the first case as H_*^γ is regular, for $\omega \in \gamma$, H_ω^γ is a complement to H_*^γ in H^γ, and maximal in H^γ. This is contrary to the argument in 6.3 in [7]. Thus H^γ is doubled, and $W(\gamma)^\gamma = J^\gamma$ by 9.1.3. Thus

$$\mathrm{Aut}_J(W(\gamma)) = \mathrm{Inn}(W(\gamma)).$$

Claim J is contained in no complement R to W_0 in W_*. Assume otherwise and set $m := |\Gamma|$. Then $R \cong S_m$ and $N_R(\gamma) \cong S_{m-1}$. But as

$$\mathrm{Aut}_J(W(\gamma)) = \mathrm{Inn}(W(\gamma)),$$

it follows that $\mathrm{Inn}(W(\gamma)) \le \mathrm{Aut}_R(W(\gamma))$. Thus as $N_R(\gamma)$ is almost simple, $N_R(\gamma)$ is faithful on $W(\gamma) \cong L$, and $L \cong F^*(N_R(\gamma)) \cong A_{m-1}$. But there is an involution $r \in R - E(R)$ inducing an outer automorphism on $E(R)$, and hence also on $W(\gamma)$. But then as $H_*^\gamma W(\gamma)^\gamma$ is doubled, r also induces an outer automorphism on H_*, contradicting $r \in W_* \le C_G(H_*)$. This completes the proof of the claim.

Next $W_0 = (W_*)_\Gamma$ and $W_0 \le D$, so $J \cap W_0 \le J \cap D = 1$ by 9.1.1. Thus by the claim, $W_* \neq W_0 J = \bar{W}_0$, so that $\bar{W}_0 \in \mathcal{W}_0$ and $\bar{W}_0 < W_*$. Therefore by 9.1.4

and 3.4, there exists $U \in \mathcal{W}_0'$ with $W_* = W_0U$ and $\bar{W}_0 \cap U = J$. Thus $J \leq U$, and $W_0 \cap U = W_0 \cap \bar{W}_0 \cap U = W_0 \cap J = 1$. Hence U is a complement to W_0 in W_* containing J, contrary to the claim. This completes the proof of (1).

Let $\mathcal{O}$ be the set of orbits of W_* on Ω. Suppose (2) fails. As H^γ is primitive, either $W_*^\gamma = 1$ or W_γ^* is transitive, and by (1) it is the former. Thus W_* is not transitive on Ω. Therefore $\mathcal{O} \in \mathcal{P}'(H)$, so $\mathcal{O} \leq \Gamma$ by 8.1.2. This is impossible as we just saw $W_*^\gamma = 1$. Thus (2) holds. Then as $J \leq W_*$ by 9.1.3, (2) implies (3).

As $J = 1$ and $H_*J/J = F^*(H/J)$, (4) holds. □

9.3. *Let $W \in \mathcal{W}_0'$ and $Y := WH$. Then:*

(1) *H is a complement to W in Y.*

(2) $\ker_H(Y) = 1$.

Proof. As $W \in \mathcal{W}_0$, $H \cap W = J = 1$ by 9.2.3. Thus (1) holds. If $1 \neq \ker_H(Y)$, then $H_* \leq \ker_H(Y)$ by 9.2.4. But then

$$[H_*, W] \leq H \cap W = 1$$

by (1), so $W \leq W_* = 1$ by 9.2.2, a contradiction. Hence (2) holds. □

9.4. *Let $W \in \mathcal{W}_0^*$ and $Y := WH$. Then:*

(1) *H is maximal in Y.*

(2) *Y is faithful and primitive on Y/H.*

(3) *$W = F^*(Y) \cong E_{p^e}$ for some prime p and integer $e > 1$, W is regular on Y/H, and Y is affine on Y/H.*

Proof. As $W \in \mathcal{W}_0^*$, part (1) follows from 2.12.1 in [6]. Then (2) follows from (1) and 9.3.2. As $W \trianglelefteq Y$, it follows from (2) that W is transitive on Y/H, and then W is regular on Y/H by 9.3.1. As W is regular on Y/H, Y is either affine or complemented, and we may assume the latter. Then W is the direct product of the set $\mathcal{C}$ of the r components of Y, and H is faithful and transitive on $\mathcal{C}$, so $H \leq S_r$. Let c be the order of a component of Y; thus $n = c^r$. Pick a prime divisor p of c. As H is transitive on Ω, n divides $|H|$, so $|H|_p \geq c_p^r \geq p^r$. But by 3.3 in [4],

$$|S_r|_p \leq p^{(r-1)/(p-1)} < p^r,$$

a contradiction. □

Theorem 9.5. *Assume Hypothesis* 5.1. *If H is transitive on Ω, then H is primitive on Ω.*

Proof. Assume otherwise. Then Hypothesis 6.1 is satisfied. Thus we may appeal to the results in this section.

Let $W \in \mathcal{W}_0^*$ and $Y := HW$. By 9.4.3, W is the unique minimal normal subgroup of Y and $W \cong E_{p^e}$ for some prime p and integer e.

If Y is primitive on Ω, we set $\Phi := \Omega$, $K := H$, and $X := Y$. Suppose instead that Y is imprimitive on Ω. Then by 8.1.4, there is a greatest member Σ of $\mathcal{P}'(Y)$, and $\Sigma \leq \Gamma$. As $\Sigma \leq \Gamma$, $H_* \leq G_\Sigma$. As H is irreducible on W,

$$W = [W, H_*] \leq G_\Sigma.$$

Thus by maximality of Σ, Σ is the set of orbits of W on Ω. In this case pick $\Phi \in \Sigma$ and set $K := N_H(\Phi)$ and $X := WK$. By construction, $H_* \leq K \leq H$, so by 9.2.4, K is almost simple with $F^*(K) = H_*$. As H is transitive on Ω and preserves Σ, K is transitive on Φ.

Thus in either case, K is almost simple with $F^*(K) = H_*$, K is transitive on Φ, and $|\Phi|$ is a power of p. Therefore by 1.5, either H_* is transitive on Φ or

$$C_A(H_*) \neq 1.$$

The latter contradicts 9.2.2, so the former holds. However as Γ is a nontrivial partition of Ω, and the set of orbits of H_* on Ω, H_* is not transitive on Ω. Thus Y is imprimitive on Ω and $\Phi \in \Sigma$. Moreover as H_* is transitive on Φ, $\Phi \in \Gamma$, so $\Sigma = \Gamma$, and hence $W \leq M$. But now by 3.3, $G = \langle \mathcal{W}_0^*, H \rangle \leq M(\Gamma)$, contrary to Hypothesis 5.1. □

10 The intransitive case

In this section we assume:

Hypothesis 10.1. Hypothesis 5.1 holds and H is intransitive on Ω.

Notation 10.2. Let $\hat{G} := F^*(G)J$, $I := \{1, \ldots, k\}$, and $\{\Omega_i : i \in I\}$ the orbits of H on Ω. For $\gamma \subseteq I$, set $\gamma' := I - \gamma$,

$$\Omega_\gamma := \bigcup_{i \in \gamma} \Omega_i, \quad X_\gamma := \hat{G}_{\Omega - \Omega_\gamma}, \quad T_\gamma := S_{\Omega - \Omega_\gamma}, \quad G_\gamma := \prod_{i \in \gamma} X_i,$$

and

$$S_I := \prod_{i \in I} T_i.$$

Recall the definition of H_* from 2.11, and set $J_* := J \cap H_*$. Set

$$I_* := \{i \in I : \Omega_i \subseteq \mathrm{Mov}(H_*)\}.$$

Observe $G_I = X_1 \times \cdots \times X_k$, and let $\pi_i : G_I \to X_i$ be the ith projection map. As H acts on each Ω_i, and $|S_I : G_I|$ is of exponent 2, $O^2(H) \leq G_I$.

For $Y \in \mathcal{I}_G(H)$, set $\bar{Y} := YJ$. Let $\mathcal{S}$ be the set of subsets γ of I such that $I_* \not\subseteq \gamma$.

10.3. *The following hold:*

(1) $H_* \leq G_{I_*}$.

(2) *For each $i \in I_*$, set $H_i := H_*\pi_i$ and $J_i := J_*\pi_i$. Then*

$$H_i/J_i \cong L,$$

so $|\Omega_i| \geq 5$.

(3) *For $\gamma \in \mathcal{S}$, $\bar{X}_\gamma$ and $\bar{G}_\gamma$ are in $\mathcal{W}_0$.*

(4) *Assume $\alpha_j \in \mathcal{S}$ for $j = 1, 2$ such that $\alpha_2 \not\subseteq \alpha_1$, $|\alpha_2| > 1$, and $|\Omega_{\alpha_2}| > 2$. Then $\bar{X}_{\alpha_1} \neq \bar{X}_{\alpha_2}$ and $\bar{X}_{\alpha_2} \in \mathcal{W}_0'$.*

(5) *For $i \in I_*$, $X_i \not\leq J$.*

(6) *Suppose $\alpha_j \in \mathcal{S}$, $j = 1, 2$, and there exists $i \in \alpha_2 \cap I_* - \alpha_1$. Then $\bar{G}_{\alpha_1} \neq \bar{G}_{\alpha_2}$.*

Proof. If H_* is nontrivial on Ω_i, then as H is transitive on Ω_i and $H_* \trianglelefteq H$, we have $i \in I_*$. Hence $H_* \leq S_{I_*}$. Then as $H_* = O^2(H_*)$, (1) follows from a remark in 10.2.

For $i \in I_*$, $\Omega_i \subseteq \mathrm{Mov}(H_*)$, so $H_i \neq 1$. If $H_i \leq J_i$, then for $x \in H_* - J_*$, there is $y \in J_*$ with $x\pi_i = y\pi_i$. Thus $xy^{-1} \in \ker(\pi_i)$, so

$$\ker(\pi_i) \cap H_* \not\leq J_*.$$

Hence as $\ker(\pi_i) = G_{i'}$ is H-invariant, we get $H_* \leq \ker(\pi_i)$ by 2.11, contradicting $H_i \neq 1$. Therefore $H_i \not\leq J_i$, so the induced map $\zeta : H_*/J_* \to H_i/J_i$ defined by $\zeta : hJ_* \mapsto (h\pi_i)J_i$ is nontrivial. Then as $H_*/J_* \cong L$ is simple, (2) holds.

Pick γ as in (3) and let $Y \in \{G_\gamma, X_\gamma\}$. For $i \in \gamma$, Ω_i is H-invariant, so Y is H-invariant. Thus if $\bar{Y} \notin \mathcal{W}_0$, then we get $H_* \leq Y$ by 2.12. But as $\gamma \in \mathcal{S}$, there is $i \in I_* - \gamma$, and by (2), $H_*\pi_i \neq 1$, a contradiction. Therefore (3) holds.

Assume the hypothesis of (4). Then X_{α_1} acts on Ω_i for $i \in \alpha_2 - \alpha_1$, but as $|\Omega_{\alpha_2}| > 2$, X_{α_2} is transitive on Ω_{α_2}, and then as $|\alpha_2| > 1$, X_{α_2} does not act on Ω_i. Thus as J acts on Ω_i, $\bar{X}_{\alpha_1} = X_{\alpha_1}J \neq X_{\alpha_2}J = \bar{X}_{\alpha_2}$, and $\bar{X}_{\alpha_2} \neq J$. Therefore (4) holds.

Suppose $i \in I_*$ with $X_i \leq J$, and let $\gamma = I_* - \{i\}$. Then by (1),

$$H_* \leq G_{I_*} = G_\gamma X_i \leq G_\gamma J = \bar{G}_\gamma,$$

contrary to (3). Thus (5) holds.

Assume the hypothesis of (6) and $\bar{G}_{\alpha_1} = \bar{G}_{\alpha_2}$. By hypothesis there exists an $i \in I_* \cap \alpha_2 - \alpha_1$. Therefore $G_i \le \bar{G}_{\alpha_2} = \bar{G}_{\alpha_1} = G_{\alpha_1} J \le G_{i'} J$. But by (1),

$$H_* \le G_{I_*} \le G_{i'} G_i \le G_{i'} J = \bar{G}_{i'},$$

whereas $\bar{G}_{i'} \in \mathcal{W}_0$ by (3). □

10.4. *Suppose $\gamma \in \mathcal{S}$ and $\alpha_j \subseteq \gamma$ for $j = 1, 2$, such that $|\Omega_{\alpha_1}| = |\Omega_{\alpha_2}| = m > 0$ and $\alpha_1 \cap \alpha_2 = \varnothing$. Then $m = 1$, $|\gamma| \le 3$ with $|\Omega_i| = 1$ for $i \in \gamma - (\alpha_1 \cup \alpha_2)$, and $\hat{G} = A$.*

Proof. By 10.3.3, $\bar{X}_\gamma \in \mathcal{W}_0$. Let $Y := N_{X_\gamma}(\{\Omega_{\alpha_1}, \Omega_{\alpha_2}\})$ and $X := N_Y(\Omega_{\alpha_1})$. Then X and Y are H-invariant, so $\bar{X}, \bar{Y} \in \mathcal{W}_0$ by 2.7. Also $|Y : X| \le 2$, and as J acts on Ω_{α_1}, if $Y \ne X$, then $\bar{Y} \ne \bar{X}$. Therefore by 3.5.2, $Y = X$. The lemma follows. □

10.5. $k \le 3$.

Proof. Assume $k > 3$. Then we may choose notation so that some member of I_* is not contained in $I_0 = \{1, 2, 3\}$, and hence $I_0 \in \mathcal{S}$. Let $I_1 = \{i \in I_0 : |\Omega_i| = 1\}$, and let I_2 be the set of 2-subsets of I_0. Set $X := X_{I_0}$.

Assume $|\Omega_\alpha| > 2$ for each $\alpha \in I_2$. Then by 10.3.4 applied to α, i_α in the role of α_2, α_1, with $i_\alpha \in \alpha$, $\bar{X}_\alpha \in \mathcal{W}_0'$ and $\bar{X}_\alpha \ne \bar{X}_{i_\alpha}$. Similarly by 10.3.4, $\bar{X} \in \mathcal{W}_0$ and $\bar{X}_\alpha \ne \bar{X}$. Observe that for distinct α and $\beta \in I_2$,

$$X = \langle X_\alpha, X_\beta \rangle \quad \text{and} \quad X_\alpha \cap X_\beta = X_{\alpha\cap\beta}.$$

Applying 3.4 to the connected component $\mathcal{C}$ of Λ' containing $\bar{X}$, it follows that $\bar{X} = L_{\mathcal{B}}$ for some $\mathcal{B} \subseteq \mathcal{C}^*$, $\bar{X}_\alpha = L_{\nu(\alpha)}$ for some $\nu(\alpha) \subseteq \mathcal{B}$, and

$$\bar{X}_{\alpha\cap\beta} = L_{\nu(\alpha)\cap\nu(\beta)}.$$

Then by 3.4 and 3.7, $\nu(\alpha) \cup \nu(\beta) = \mathcal{B}$ and $\nu(\alpha \cap \beta) = \nu(\alpha) \cap \nu(\beta)$. But now

$$\mathcal{B} = \nu(\{1, 2\}) \cup \nu(\{1, 3\}) \cup \nu(\{2, 3\}),$$

so as $X_{i,j} \cap X_{i,l} = X_i$, it follows from 3.4 that $X = Y$, where $Y = \langle X_1, X_2, X_3 \rangle$. As Y acts on Ω_i for each $1 \le i \le 3$, but X does not, this is a contradiction.

Therefore $|\Omega_\alpha| = 2$ for some $\alpha \in I_2$. We may take $\alpha = \{1, 2\}$, so $|\Omega_i| = 1$ for $i \in \{1, 2\}$. Therefore by 10.4, $|\Omega_j| = 1$ for each $j \in I_0$, and $\hat{G} = A$. Since we could have chosen I_0 to be any 3-subset of I not contained in I_*, and since $|\Omega_i| \ge 5$ for each $i \in I_*$ by 10.3.2, it follows that $|I_*| = 1$ and $|\Omega_i| = 1$ for each $i \in I_*'$. Finally if $k > 4$, we obtain a contradiction from 10.4 by choosing $\alpha_j = \{j\}$ for $j = 1, 2$ and $\gamma = I_*'$ in that lemma. Thus we have shown that $k = 4$, and we may choose notation so that $I_* = \{4\}$ and $|\Omega_i| = 1$ for $1 \le i \le 3$.

Now as $\hat{G} = A$, $X \cong \boldsymbol{Z}_3$. Further all orbits of H on $\Omega_{1,2,3}$ are of length 1, so H centralizes X and $J \cap X = 1$. Therefore $J \neq \bar{X} = N_{\bar{X}}(H)$, contrary to 3.5.1. This completes the proof of the lemma. □

10.6. *Assume* $\{1, 2\} \in \mathcal{S}$ *and* $\bar{X}_2 \in \mathcal{W}_0'$. *Then* $\bar{X}_2 = \bar{G}_{1,2}$, *so* $X_1 \leq J$.

Proof. Assume otherwise. By 10.3.3, $\bar{X}_{1,2}$, $\bar{X}_j$, $j = 1, 2$, and $\bar{G}_{1,2}$ are in $\mathcal{W}_0$. By hypothesis, we have $\bar{X}_2 \in \mathcal{W}_0'$ and $\bar{X}_2 \neq \bar{G}_{1,2}$, so also $\bar{X}_1 \in \mathcal{W}_0'$. Let $m_j = |\Omega_j|$. As $\bar{X}_j \in \mathcal{W}_0'$, $m_j > 2$ by 3.5.3. By 10.4, $m_1 \neq m_2$, so we may assume $m_2 > m_1$.

We apply 3.4 and 3.7 to $X_{1,2}$ in the role of Y, $X_{1,2}H$ in the role of X, and $\mathcal{C}$ the connected component of Λ' containing $\bar{X}_{1,2}$. In particular $\bar{X}_{1,2} = L_{\mathcal{A}}$ for some $\mathcal{A} \subseteq \mathcal{C}^*$, $\bar{X}_i = L_{\alpha_i}$ for some $\alpha_i \subseteq \mathcal{A}$, and $\bar{G}_{1,2} = L_\gamma$, where $\gamma = \alpha_1 \cup \alpha_2 \subset \mathcal{A}$ as $\bar{G}_{1,2} \neq \bar{X}_{1,2}$. Pick $a \in \mathcal{A} - \gamma$, and set

$$\beta := \alpha_2 \cup \{a\}, \quad U := L_\beta, \quad \text{and} \quad Z := X_{1,2} \cap U.$$

By 3.7, $U = \bar{Z}$, and as $\beta \subset \mathcal{A}$, $Z \neq X_{1,2}$. As $a \notin \gamma$, it follows that $Z \not\leq G_{1,2}$, so Z_H is transitive on $\Omega_{1,2}$. As $m_2 > 2$, X_2 contains a 3-cycle, so as $Z \neq X_{1,2}$ and ZH is transitive on $\Omega_{1,2}$, it follows from 3.2 in [8] that Z is imprimitive on $\Omega_{1,2}$, and Ω_2 is contained in a block of each nontrivial Z-invariant partition of $\Omega_{1,2}$. As $m_2 > m_1 = |\Omega_{1,2} - \Omega_2|$, this is a contradiction. □

10.7. *Assume* $k = 3$. *Then:*

(1) $I_*' \neq \varnothing$, *and*

(2) *for some* $i \in I_*'$, $X_i \leq J$.

Proof. Assume otherwise. We may choose $I_0 = \{1, 2\} \in \mathcal{S}$. Thus for $i \in I_0$ we have $\bar{X}_i \in \mathcal{W}_0$ by 10.3.3. If also $i \in I_*$, then $\bar{X}_i \in \mathcal{W}_0'$ by 10.3.5. Therefore by 10.6, there exists $j \in I_0 \cap I_*'$ such that $X_j \leq J$, completing the proof. □

Theorem 10.8. *Assume* $k = 3$ *and choose notation so that* $I_* = \{3\}$ *or* $\{2, 3\}$. *If* $I_* = \{3\}$, *assume* $X_2 \neq 1$. *Then:*

(1) $|\Omega_1| \neq |\Omega_2|$, *and*

(2) $N_{X_{1,2}}(\Omega_1) \leq J$.

We prove Theorem 10.8 in a series of reductions, so assume the hypotheses of the theorem. For $1 \leq i \leq 3$, set $m_i := |\Omega_i|$, and set $M := N_{X_{1,2}}(\Omega_1)$.

If $I_* = \{3\}$, then $X_2 \neq 1$ by hypothesis, while if $I_* = \{2, 3\}$, this follows from 10.3.2. Thus

10.9. $X_2 \neq 1$.

By 10.9, $m_2 > 1$, so 10.8.1 follows from 10.4. By 10.8.1 and 3.7.1 in [8]:

10.10. *M is a maximal subgroup of $X_{1,2}$.*

In the remainder of the proof of Theorem 10.8, assume $M \not\leq J$.

10.11. *The following hold:*

(1) $\bar{M}, \bar{X}_{1,2} \in \mathcal{W}_0'$ *with* $\bar{M} < \bar{X}_{1,2}$.

(2) *There exists a unique* $U \in \mathcal{W}_0^*$ *contained in* $\bar{X}_{1,2}$ *but not in* $\bar{M}$. *Set*

$$U_X := U \cap X_{1,2}.$$

(3) HU_X *is transitive on* $\Omega_{1,2}$.

Proof. By 10.3.3, we have $\bar{X}_{1,2} \in \mathcal{W}_0$. Hence as $M \in \mathcal{I}_{X_{1,2}}(H)$, $\bar{M} \in \mathcal{W}_0$ by 2.7, so as $M \not\leq J$, $\bar{M} \in \mathcal{W}_0'$. As M acts on Ω_1 but $X_{1,2}$ does not, $\bar{M} < \bar{X}_{1,2}$ by an argument in the proof of 10.3.4, so (1) holds.

As M is maximal in $X_{1,2}$, $\bar{M}$ is maximal in $\mathcal{W}_0(< \bar{X}_{1,2})$ by 3.7, so (2) follows from 3.4. By maximality of M, $X_{1,2} = \langle U_X, M\rangle$, and in particular $U_X \not\leq M$. Then as H is transitive on Ω_i, (3) follows. □

Let $\mathcal{C}$ be the connected component of Λ' containing $\bar{X}_{1,2}$ and $\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*$. By 3.8 some $V \in \mathcal{C}^*$ does not act on $\Omega_{1,2}$. Let

$$U_X \leq Y \in \mathcal{I}_{X_{1,2}}(H)$$

with $\bar{Y} \neq \bar{X}_{1,2}$, and set $Z_Y := \langle Y, V\rangle$ and $B_Y := Z_Y H$.

10.12. *The following hold:*

(1) $Z_Y \in \mathcal{C}$, $\bar{Y} \cap V = J$, *and* $\bar{X}_{1,2} \cap Z_Y = \bar{Y}$.

(2) B_Y *is transitive on* Ω.

(3) $A \not\leq B_Y$.

Proof. Part (1) follows from 3.4. As UH is transitive on $\Omega_{1,2}$ and Ω_3, and as V does not act on $\Omega_{1,2}$, (2) holds. As $Z_Y \in \mathcal{C}$, (3) holds.

By 10.7, $X_i \leq J$ for $i = 1, 2$. Choose such an $i \in \{1, 2\}$. □

10.13. *Assume $X_i \neq 1$.*

(1) $\Delta_Y := \Omega_1^Y$ *is a* UH*-partition of* $\Omega_{1,2}$, YH *is 2-transitive on* Δ_Y, Y *is transitive on* $\Omega_{1,2}$, *and* $K_+(\Delta_Y) \leq Y$.

(2) *We may assume* $i = 1$.

(3) $m_1 < m_2$.

Proof. Set $j = 3-i$. As $X_{1,2} \not\leq Y$, 3.2 in [8] says that there is a YH-partition Δ on $\Omega_{1,2}$ such that $\Omega_i \subseteq \delta \in \Delta$ and $K_+(\Delta) \leq Y$. As H is transitive on $\Omega_{1,2} - \Omega_i$, it follows that $\Omega_i = \delta$ and YH is 2-transitive on Δ. In particular Y is transitive on Δ and hence also on $\Omega_{1,2}$, and $m_i \leq m_j$. This proves (1), and $m_i < m_j$ by 10.8.1.

By symmetry between i and j, $X_j \not\leq J$. Thus by 10.7.2, $i \in I'_*$. Thus by the hypothesis of Theorem 10.8, either $i = 1$ or $I_* = \{3\}$. Moreover in the latter case, interchanging 1 and 2 if necessary, we may assume $i = 1$. Thus $m_1 < m_2$. □

10.14. *We may assume $m_1 < m_2$.*

Proof. If $X_i \neq 1$, this follows from 10.13, so assume otherwise. Thus $m_i \leq 2$. If $I_* = \{2, 3\}$, then $m_2 \geq 5$ by 10.2.3, so $i = 1$ and the lemma holds. If $I_* = \{3\}$, interchanging 1 and 2 if necessary, and appealing to 10.8.1, the lemma holds. □

10.15. *V acts on Ω_2.*

Proof. Set $W = \langle M, V\rangle$ and assume V does not act on Ω_2. Then either W is transitive on V or WH has orbits $\{\Omega_2, \Omega_{1,3}\}$ or $\{\Omega_1, \Omega_{2,3}\}$, so we may assume W is transitive on $\Psi = \Omega$ or $\Omega_{2,3}$. As $H_* \leq X_{2,3}$, we have $X_{2,3} \not\leq W$. On the other hand $1 \neq X_2 \leq M \leq W$, so by 3.2 in [8] there is a nontrivial WH-invariant partition Σ of Ψ such that $\Omega_2 \subseteq \sigma \in \Sigma$ and $K_+(\Sigma) \leq W$. As H is transitive on Ω_1 and Ω_3, and as $m_1 < m_2$ by 10.14, it follows that either $\sigma = \Omega_2$ or $\Psi = \Omega$ and $\sigma = \Omega_{1,2}$. In the latter case, $X_{1,2} \leq K_+(\Sigma) \leq W$, so $U \leq W$, contrary to 3.4. Thus $\Omega_2 = \sigma$.

Next for $\omega \in \Omega_1$, we have $\omega \in \alpha \in \Sigma$ and $\omega K_+(\Sigma) = \alpha$ is of order $m_2 > m_1$, so $K_+(\Sigma)$ does not act on Ω_1. Thus $K_+(\Sigma) \not\leq \bar{M}$, so $V \leq K_+(\Sigma)J$. But then V acts on Ω_2, contrary to assumption. □

10.16. *$\Omega_{1,2}^{B_Y}$ is not a partition of Ω.*

Proof. Assume otherwise. First, Y acts on $\Omega_{1,2}$, and by 10.15, V acts on Ω_2, so V also acts on $\Omega_{1,2}$. But then $B_Y = \langle V, Y, H\rangle$ acts on $\Omega_{1,2}$, contrary to 10.12.2. □

10.17. $X_1 = 1$.

Proof. Assume otherwise. Applying 10.13 to U in the role of Y, we obtain the UH-partition $\Delta = \Delta_U = \Omega_1^U$. By 10.13.1, we have $K_+(\Delta) \leq U \leq Y$, so as Δ is the unique $K_+(\Delta)$-invariant partition of $\Omega_{1,2}$ with blocks of size m_1, we conclude that $\Delta = \Delta_Y$ is independent of Y. In particular $N_{X_{1,2}}(\Delta)$ is the unique maximal overgroup of U_X in $X_{1,2}$, so it follows from 3.4 that $\bar{M} \in \mathcal{W}_0^*$. Then applying 3.7 with $X_{1,2}, X_{1,2}H$ in the roles of Y, X, we conclude that U_X is a maximal H-invariant subgroup of $X_{1,2}$. Thus $U_X = N_{X_{1,2}}(\Delta)$ and $Y = U_X$. Set $Z = Z_Y$ and $B = B_Y$.

By 10.12.2, B is transitive on Ω, and as $H_* \not\leq Z$, we have $A \not\leq B$. Therefore by 3.2 in [8], there is a B-partition Γ of Ω with $\Omega_1 \subseteq \gamma \in \Gamma$ and $K_+(\Gamma) \leq Z$. As H has orbits Ω_k, $1 \leq k \leq 3$, it follows that $\gamma = \Omega_1, \Omega_{1,2}$, or $\Omega_{1,3}$. As $\bar{Z} \cap \bar{M} = J$, $M \not\leq Z$, so as $K_+(\Gamma) \leq Z$, $\gamma \neq \Omega_{1,2}$.

Suppose $\gamma = \Omega_{1,3}$ and let $u \in U$ with $\Omega_1 u \neq \Omega_1$. Then $\gamma u \neq \gamma$ so $\gamma u \subseteq \Omega_2$, and then as $X_{1,3} \leq K_+(\Gamma) \leq Z$, we have $X_{1,3}^u \leq Z \cap X_2 = Y$, impossible as $X_{1,3}^u$ does not preserve Δ. Therefore $\gamma = \Omega_1$, so $\Gamma = \Omega_1^Z$. In particular $\Delta \subseteq \Gamma$.

Next $Y^\Delta = \mathrm{Sym}(\Delta)$ and H is transitive on $\Gamma - \Delta$, so if B is imprimitive on Γ, then Δ^B is a partition of Γ. But then $\Omega_{1,2}^B$ is a partition of Ω, contrary to 10.16.

Therefore B is primitive on Γ. Let K be the kernel of the action of B on Γ. Then $K/K_+(\Gamma)$ is solvable and $K_+(\Gamma) \leq Z$, so $\bar{K} \in \mathcal{W}_0$ by 2.8. Then $K \leq Z$ by 3.4. Also as $Y^\Delta = \mathrm{Sym}(\Delta)$, 3.2 in [8] says $\mathrm{Sym}(\Gamma) = B^\Gamma$, contrary to 2.15. □

10.18. $m_1 \leq 2$.

Proof. This follows from 10.17 and 10.9. □

10.19. *If Y is transitive on $\Omega_{1,2}$, then B_Y is imprimitive on Ω.*

Proof. Assume otherwise. By 1.7.3, $B := B_Y$ is 2-transitive on Ω, so by 2.15, B is affine and $D = F^*(B) \leq Z$. Thus $\bar{D} \in \mathcal{W}_0$ by 2.7. We apply 3.7 to D, B in the roles of Y, X, and adopt the notation of that lemma. As $D \not\leq J$, it follows that $\mathcal{A} \neq \varnothing$ and $DJ = \bar{D} = L_{\mathcal{A}}$. Let $a \in \mathcal{A}$; then $1 \neq D_a = L_a \cap D$ is JD-invariant as D is abelian, and by 3.7, D_a is $L_{\mathcal{B}}$-invariant, so

$$D_a \trianglelefteq \langle L_{\mathcal{A}}, L_{\mathcal{B}}, H \rangle = B$$

using 3.4. Therefore as B is irreducible on D, $D = D_a$, so $\mathcal{A} = \{a\}$. As Y acts on Ω_2, $L_a \not\leq \bar{Y}$, so $L_a = V$. Thus $B = \langle Y, V, H \rangle = \bar{Y} D$. But now 1.7.4 supplies a contradiction. □

10.20. $m_1 = 2$.

Proof. Assume otherwise and let $B := B_Y$. Then $m_1 = 1$ by 10.18. By 10.11.3, HY is transitive on $\Omega_{1,2}$, so as H is transitive on Ω_2, HY is 2-transitive on $\Omega_{1,2}$. By 10.19, B is imprimitive on Ω. Therefore by 1.7.2, $\Gamma = \Omega_{1,2}^B$ is a partition of Ω, contradicting 10.16. □

10.21. *The following hold:*

(1) *There exists a nontrivial YH-partition Δ on $\Omega_{1,2}$.*

(2) *Either $\Omega_1 \in \Delta$ or H_{Ω_1} has two orbits $\theta, \theta h$ of length $m_2/2$ on Ω_2, for $h \in H - H_{\Omega_1}$, and $\Delta = \{\Xi, \Xi h\}$, where $\Xi = \{\omega\} \cup \theta$ for some $\omega \in \Omega_1$.*

(3) *We may choose* $Y = N_{X_{1,2}}(\Delta)$, *so* Y *is transitive on* $\Omega_{1,2}$.

(4) *If* $Y = N_{X_{1,2}}(\Delta)$, *then there is a nontrivial* B_Y*-partition* Γ *of* Ω *with* $\Delta \subseteq \Gamma$.

Proof. Set $B := B_Y$. Suppose that (1) fails. By 1.6 applied to the action of YH on $\Omega_{1,2}$, with Ω_1 in the role of Σ, YH is 2-transitive on $\Omega_{1,2}$. Then by 1.7.1, Y is transitive on $\Omega_{1,2}$, so by 10.19, B is imprimitive on Ω. Then 1.7.2 applied to the action of B on Ω, $\Omega_{1,2}^B$ is a partition of Ω, contrary to 10.16.

This establishes (1). Then (2) follows from (1) and 1.6. As HU acts on Δ, (3) follows.

Assume $Y = N_{X_{1,2}}(\Delta)$. By (3) and 10.19 there is a nontrivial B-partition Γ on Ω. Let $\gamma \in \Gamma$ with $\gamma \cap \Omega_{1,2} \neq \varnothing$. By 1.1, either $\Omega_{1,2} \subseteq \gamma$ or $\gamma \subseteq \Omega_{1,2}$. In the former case as H is transitive on Ω_3, $\gamma = \Omega_{1,2}$, contrary to 10.16. Therefore $\Delta_1 = \{\gamma \in \Gamma : \gamma \subseteq \Omega_{1,2}\}$ is a nontrivial YH-partition of $\Omega_{1,2}$, so as Δ is the unique nontrivial YH-partition of $\Omega_{1,2}$, $\Delta_1 = \Delta$. Therefore $\Delta \subseteq \Gamma$, completing the proof of (4). □

Choose Γ and Δ as in 10.21 and $Y = N_{X_{1,2}}(\Delta)$. Let $\gamma = \Omega_1$ or Ξ in the respective case of 10.21.2; thus $\gamma \in \Gamma$. By construction:

10.22. $\Gamma - \Delta \subseteq \mathrm{Fix}(Y)$ *and* $Y^\Delta = \mathrm{Sym}(\Delta)$.

Set $B := B_Y$ and $Z := Z_Y$. By10.22 we can apply 1.7 to the action of B^Γ on Γ. Then if B^Γ is imprimitive, we conclude from 1.7.2 that Δ^B is a B-partition of Γ. But then $\Omega_{1,2}^B$ is a B-partition of Ω, contrary to 10.16.

Therefore B^Γ is primitive, so it follows from 10.22 and 3.2.1 in [8] that

$$B^\Gamma = \mathrm{Sym}(\Gamma).$$

We claim that $B_\Gamma \leq Z$. If so, then as $Y^\Gamma \neq 1$, $B_\Gamma < Z$, so as $B^\Gamma = \mathrm{Sym}(\Gamma)$, 2.15.1 supplies a contradiction. Thus it remains to establish the claim. But Y contains $K_+(\Delta)$, so as B^Γ is transitive, Z contains $K_+(\Gamma)$. Then as $B_\Gamma/K_+(\Gamma)$ is a 2-group, appealing to 2.13 and enlarging Z to ZB_Γ if necessary, we have $B_\Gamma \leq Z$, establishing the claim. This completes the proof of Theorem 10.8.

10.23. *Assume* $k = 3$. *Then:*

(1) $|I_*| = 1$, *so we may choose notation so that* $I_* = \{1\}$.

(2) *Assume* $|\Omega_{2,3}| > 2$, *and set* $M := N_{X_{2,3}}(\Omega_2)$. *Then*

$$M = X_{2,3} \cap J \quad \textit{and} \quad \bar{X}_{2,3} \in \mathcal{W}_0^*.$$

Proof. By 10.7.1, $|I_*| = 1$ or 2. If $|I_*| = 2$, then we may take $I_* = \{2, 3\}$, so by 10.8, $X_2 \leq J$, contrary to 10.3.5. Thus (1) is established and we choose $I_* = \{1\}$.

It remains to prove (2), so we may assume $m := |\Omega_{2,3}| > 2$. Suppose $X_i = 1$ for $i = 2, 3$. Then $m_i = |\Omega_i| \leq 2$ for $i = 2, 3$, so as $m > 2$, we may assume $m_i = 2$ for some $i \in \{2, 3\}$. Then as $X_i = 1$, $\hat{G} = A$. It follows that either $m = 3$ and the subgroup M of (2) is trivial, or $m = 4$. In the former case, $\boldsymbol{Z}_3 \cong X_{2,3}$, so $\bar{X}_{2,3} \in \mathcal{W}_0^*$ by 2.8, and hence (2) holds. Thus we may assume the latter case holds. But now $m_2 = m_3 = 2$, and hence 10.4 supplies a contradiction.

Thus we may assume that $X_2 \neq 1$. Then, modulo change of notation, the hypotheses of 10.8 are satisfied, and we conclude from that theorem that $M \leq J$, and $m_2 \neq m_3$, so M is maximal in $X_{2,3}$ (cf. 3.7.1 in [8]). By 10.3.4, $\bar{X}_{2,3} \in \mathcal{W}_0'$, so it follows from the maximality of M in $X_{2,3}$ and 3.7 that

$$M = X_{2,3} \cap J \quad \text{and} \quad \bar{X}_{2,3} \in \mathcal{W}_0^*.$$

This completes the proof of the lemma. □

Hypothesis 10.24. $|I_*| = 1$.

Notation 10.25. When assuming Hypothesis 10.24, we adopt the following notation: Take $I_* := \{1\}$ and let $X := X_{1'}$ and $\Omega_X := \Omega_{1'}$. Observe that Hypothesis 10.24 is always satisfied when $k = 3$ by 10.23.1, so we always adopt Notation 10.24 when $k = 3$.

10.26. *Assume $k = 3$. Then exactly one of the following holds:*

(1) $|\Omega_X| > 2$ *and* $\bar{X} \in \mathcal{W}_0^*$.

(2) $|\Omega_X| = 2$, $\hat{G} = A$, *and* $X = 1$.

Proof. If $|\Omega_X| = 2$, then (2) holds by 10.4, so we may assume $|\Omega_X| > 2$. Then (1) holds by 10.23.2. □

10.27. *Assume Hypothesis* 10.24*, and suppose* $Y_1 \in \mathcal{J}_{X_1}(H)$ *with* $\bar{Y}_1 \in \mathcal{W}_0$. *Then:*

(1) *For each* $Y_2 \in \mathcal{J}_X(H)$, $\bar{Y}_1 Y_2 \in \mathcal{W}_0$.

(2) *If* $\bar{Y}_2 \leq \bar{Y}_1$, *then* $Y_2 \leq S_I$ *and* $Y_2 \leq N_G(U \cap X)$ *for each* $U \in \mathcal{W}$.

(3) *Assume* $Y_1 \not\leq J$ *and* $\bar{Y}_1 = \bar{X}$. *Then* $k = 2$, $\bar{X} \in \mathcal{W}_0^*$, $\hat{G} = A$, $|\Omega_X| > 2$, *and* $J \cap X = O_2(X)$.

Proof. Let $Y_2 \in \mathcal{J}_X(H)$ and set $B := Y_1 Y_2 J \cap X_1 X$. Then $B = Y_1 Y_2 J_I$, where $J_I := J \cap X_1 X \leq S_I$.

Suppose (1) fails. Then by 2.12, $H_* \leq B$. Set $B^* = B/Y_1$. Then $B^* = Y_2^* J_I^*$ and H_* centralizes Y_2 and acts on J_I, so

$$H_*^* = [H_*^*, H_*^*] \leq [H_*^*, Y_2^* J_I^*] = [H_*^*, J_I^*] \leq J_I^*.$$

Therefore $H_* \leq Y_1 J_I \leq \bar{Y}_1$, contradicting $\bar{Y}_1 \in \mathcal{W}_0$. This establishes (1).

Assume $\bar{Y}_2 \le \bar{Y}_1$. Then $Y_2 \le X_1 J \le S_I$ as $J \le S_I$. Then as $[X_1, X] = 1$ and J acts on $U \cap X$ for each $U \in \mathcal{W}$, (2) follows.

Assume the hypothesis of (3). As $Y_1 \not\le J$ and $\bar{Y}_1 = \bar{X}$, it follows that $X \not\le J$, so in particular $X \neq 1$, and hence X is transitive on Ω_X. By (2), $X \le S_I$ so as X is transitive on Ω_X, $k = 2$. By (2), $X \le N_G(U \cap X)$ for each $U \in \mathcal{W}$, so in particular $J_X := J \cap X \trianglelefteq X$. Thus either $O^2(X) \le J_X$ or $J_X = 1$, or $|\Omega_X| = 4$ and $J_X = O_2(X)$.

Suppose $O^2(X) \le J$. Then as $X \not\le J$, we have $X = T$, where $T = S_{\Omega - \Omega_X}$ and $J_X = O^2(X)$ is of index 2 in X. But then $|\bar{X} : J| = 2$, contrary to 3.5.3. Thus $O^2(X) \not\le J$. Thus $J_X = 1$ or $O_2(X)$, so $J_X \le O_2(X)$, and hence $X = O^2(X)$ by 3.5.2. Therefore $\hat{G} = A$, and then as X is transitive on Ω_X, $|\Omega_X| > 2$.

If $|\Omega_X| \neq 4$, then X is the only nontrivial X-invariant subgroup of T, so as X acts on $X \cap U$ for each $U \in \mathcal{W}$, it follows that $\mathcal{W}_0(< \bar{X}) = \varnothing$, so $\bar{X} \in \mathcal{W}_0^*$. Thus (3) holds in this case. Finally suppose $|\Omega_X| = 4$. Then

$$\mathcal{W}_0(< \bar{X}) = \{O_2(X)J, XJ\},$$

so we conclude from 3.4 that (3) holds. This completes the proof of (3). □

10.28. *Assume Hypothesis* 10.24. *Then* $\bar{X} \notin \mathcal{W}_0^*$.

Proof. Assume $\bar{X} \in \mathcal{W}_0^*$, let $\mathcal{C}$ be the connected component in Λ' of $\bar{X}$, and adopt the notation of 3.4. By 3.8 there is $V \in \mathcal{C}^*$ such that V does not act on X, and hence does not act on Ω_X. As $X \in \mathcal{W}_0^*$, $X \not\le J$, so $X \neq 1$ when $|\Omega_X| = 2$, and therefore X is transitive on Ω_X. Indeed as $X \not\le J$, $|\Omega_X| > 2$ by 3.5.3.

Set $W := \langle X, V\rangle$. As $\bar{X} \in \mathcal{W}_0^*$, we conclude from 3.4 that $W \in \mathcal{W}_0$, $\bar{X} \cap V = J$, and $\mathcal{W}_0(\le W) = \{J, \bar{X}, V, W\}$. As W does not act on Ω_X, and as H acts on W and is transitive on $\Omega - \Omega_X$, HX is transitive on Ω. Then as $X \le W$, it follows from 3.3 in [8] that W is transitive on Ω and WH preserves the partition $\Gamma := \Omega_X^W$ of Ω. As $X \le W$ and W is transitive on Ω, it follows that $K := \langle X^W\rangle \trianglelefteq W$, and either $\hat{G} = A$ and $K = K_+(\Gamma)$, or $\hat{G} = S$ and $K = S_\Gamma$. In particular, K acts on Ω_X. Also $W = \langle X, V\rangle = KV$. Then by 2.7, $\bar{X} \le \bar{K} \in \mathcal{W}_0$ with $\bar{K} \le W$, and as $\bar{K} = KJ$ acts on Ω_X but V does not, it follows that $V \not\le \bar{K}$. Therefore as $\mathcal{W}_0(< W) = \{J, \bar{X}, V\}$, we conclude that $\bar{K} = \bar{X}$.

Next $K = K_1 \times X$, where $K_1 = K \cap X_1$. As $W = KV$ is transitive on Ω, V is transitive on Γ, so $K = \langle X^V\rangle = \langle K_1^V\rangle$. If $K_1 \le J$, then $K_1 \le V$, so

$$K = \langle K_1^V\rangle \le V,$$

and hence $W = KV = V$, a contradiction. Hence $K_1 \not\le J$, so $\bar{K}_1 \in \mathcal{W}_0'$, and therefore $\bar{K}_1 = \bar{K} = \bar{X}$ as $\bar{X} \in \mathcal{W}_0^*$. Therefore by 10.27.3, $k = 2$, $\hat{G} = A$, and $O_2(X) = J \cap X = J_X$.

Next $V \cap K \leq V \cap \bar{X} = J$, so $V \cap K = J \cap K =: J_K$, and J_K is V-invariant. As $\bar{K} = \bar{K}_1$, $K = K_1 J_K$, so the projection $\pi_X : J_K \to X$ is a surjection. Then as V is transitive on Γ, $\pi_{X^w} : J_K \to X^w$ is a surjection for all $w \in W$. Therefore if $|\Omega_X| > 4$, then by 1.4 in [7], J_K is the product of full diagonal subgroups of K. Further $X \cap J = O_2(X) = 1$, so J_K is a complement to X in K, and then we conclude that $K_1 \cong J_X \cong X$ and $K_1 = X_1$. But now $H_* \leq X_1 \leq W$, a contradiction.

Therefore $|\Omega_X| = 3$ or 4. Then as $\bar{K} = \bar{X}$ and $X \cap J = O_2(X)$, it follows that $|K : J_K| = 3$. Let $Q := O_2(K)$ and $\tilde{K} := K/Q$. Thus

$$Q \leq J_K \trianglelefteq K$$

and $|\tilde{K} : \tilde{J}_K| = 3$.

As $H_* \leq X_1$, $|\Omega_1| > 4$ by 10.3.2, so $|\Gamma| > 2$. But $U = W\hat{G}_\Gamma \in \mathcal{W}_0$ by 2.13, as $\hat{G}_\Gamma / K$ is solvable and W acts on $\hat{G}_\Gamma$. Thus $U \in \mathcal{C}$, so by 3.7, U acts on J_K. Now there exists t in the kernel of the action of $\hat{G}$ on Γ inverting exactly two members of $\tilde{X}^W = \{\tilde{X}(i) : 1 \leq i \leq r\}$. Say $\langle \tilde{x}_i \rangle = \tilde{X}(i)$, and t inverts $\tilde{x}_j$ for $j \in \{1, 2\}$. Now as $\tilde{J}_K$ is a complement to $\tilde{X}(i)$ in $\tilde{K}$ for each i, $\tilde{X}(1)\tilde{X}(3) \cap \tilde{J}_K$ is a full diagonal subgroup of $\tilde{X}(1)\tilde{X}(3)$, and hence, replacing x_3 by x_3^{-1} if necessary, is generated by $\tilde{x} = \tilde{x}_1\tilde{x}_3$. As $t \in U$ and U acts on J_K, also $[\tilde{x}, t] = \tilde{x}_1$ is in $\tilde{J}_K$, contradicting $\tilde{X} \cap \tilde{J}_K = 1$. This contradiction completes the proof. □

10.29. *Either*

(1) $k = 2$, *or*

(2) $k = 3$, $|\Omega_i| = 1$ *for* $i \in \{2, 3\}$, *and* $X_{2,3} = 1$, *so* $\hat{G} = A$.

Proof. By 10.5, $k \leq 3$, and in case of equality, Hypothesis 10.24 is satisfied by 10.25. Thus the lemma follows from 10.26 and 10.28. □

10.30. *Assume Hypothesis* 10.24 *and suppose* $\bar{X} \in \mathcal{W}_0'$. *Let* $\mathcal{C}$ *be the connected component of* Λ' *containing* $\bar{X}$, $\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*$, *and*

$$\mathcal{C}_X^* := \{V \in \mathcal{C}^* : V \not\leq N_G(\Omega_X)\}.$$

Then $k = 2$ *and the following hold:*

(1) $\mathcal{W}_0(\leq \bar{X}) \cong \Delta(m)$ *for some* $m \geq 2$.

(2) *The map* $Z \mapsto \bar{Z}$ *is an isomorphism of* $\mathcal{V}_X(H)$ *with* $\mathcal{W}_0(\leq \bar{X})$.

(3) *Let* $\pi_i : T_1 \times T_2 \to T_i$ *be the projection map. Then* $X\pi_2^{-1} \cap H = H_1 \times J_X$, *where* $H_1 := H \cap T_1$ *and* $J_X := J \cap X$.

(4) J_X *is transitive on* Ω_X.

(5) $\mathcal{C}_X^* \neq \varnothing$ *and for each* $V \in \mathcal{C}_X^*$, VH *is transitive and imprimitive on* Ω.

Proof. By 10.28, $\bar{X} \notin \mathcal{W}_0^*$, so (1) follows from the assumption that $\bar{X} \in \mathcal{W}_0'$, and the fact that Λ is a $D\Delta(m_1, \dots, m_t)$-lattice. Part (2) follows from 3.7.

By (1), $X \neq 1$ and $\bar{X} \notin \mathcal{W}_0^*$, so by 10.26, $k = 2$. Therefore H is transitive on Ω_1 and $\Omega_X = \Omega_2$.

Next as H acts on Ω_i for $i = 1, 2$, $H \leq T_1 \times T_2$. Let $Y_i := H\pi_i$ be the projection of H on T_i, and $H_i := Y_i \cap X_i$. Now $H_2 \in \mathcal{V}_X(H)$, so by (2), $\bar{H}_2 \in \mathcal{W}_0$ and there exists a unique $B \in \mathcal{V}_X(H)$ with $X = \langle B, H_2 \rangle$ and $B \cap H_2 = J_X$. As T_1 centralizes T_2, $\mathcal{V}_X(H) = \mathcal{I}_X(Y_2) \cap \mathcal{O}_X(J_X)$. In particular each member of $\mathcal{V}_X(H)$ is H_2-invariant, so $B \trianglelefteq \langle B, H_2 \rangle = X$. Therefore we have $B = 1$, or $O^2(X) \leq B$, or $|\Omega_X| = 4$ and $B = O_2(X)$. In the first case $J_X = 1$ and as $X = \langle B, H_2 \rangle$, we get $H_2 = X$. Then $\mathcal{V}_X(H) = \mathcal{I}_X(X)$ is not isomorphic to $\Delta(m)$ for any $m \geq 2$, contrary to (1) and (2). In the second case, by 3.5.2, $B = X$, and then

$$J_X = B \cap H_2 = H_2,$$

so that (3) holds. In the third case $X = H_2 B = H_2 O_2(X)$, so H_2 is a complement to $O_2(X)$ in X, or $H_2 = X$. The first case is impossible as H, and hence also H_2, is transitive on Ω_X. The second is impossible as $\mathcal{V}_X(H) \cong \Delta(m)$ by (1) and (2). This completes the proof of (3).

As H is transitive on Ω_X, so is Y_2. Then as $|Y_2 : H_2| \leq 2$ with $H_2 = J_X$, either (4) holds or $|Y_2 : J_X| = 2$ and $Y_2 \not\leq J$. In the latter case $\bar{Y}_2 \in \mathcal{W}_0'$ by 2.13, contrary to 3.5.3. This establishes (4).

By 3.8, $\mathcal{C}_X^* \neq \varnothing$. Let $V \in \mathcal{C}_X^*$. As H is transitive on Ω_i for $i = 1, 2$ and V does not act on Ω_2, $Y := VH$ is transitive on Ω. Suppose Y is primitive. Then by (4) and 1.7.3, Y is 2-transitive on Ω and either almost simple or affine. Thus Y is affine by 2.14. Now Y has a unique minimal normal subgroup D, and as $1 \neq V \trianglelefteq Y$, $D \leq V$. As D is transitive on Ω but H is not, $D \not\leq J$, so $V = \bar{D}$ as $V \in \mathcal{W}_0^*$. Then

$$Y = DJH = DH.$$

But now 1.7.4 supplies a contradiction, establishing (5). □

Before we leave the setup of Lemma 10.30, we obtain a contradiction to the hypothesis of the lemma. By (5), there is $\Gamma \in \mathcal{P}'(Y)$. Pick Γ minimal subject to this constraint. Thus Y^Γ is primitive. By (4) and 1.1, Ω_X is a union of a set Γ_X of blocks of Γ, and J_X is transitive on Γ_X. As H acts on $\Gamma_X \subset \Gamma$ and Y is transitive on Ω, $V^\Gamma \neq 1$, so V^Γ is transitive. Similarly $V^\Gamma \neq J^\Gamma$, so $\bar{V} \neq \bar{V}_\Gamma$, and hence as $V \in \mathcal{W}_0^*$, $V_\Gamma = J_\Gamma$. Therefore $J_\Gamma \trianglelefteq Y$.

Let $B := VY_\Gamma$ be the preimage of V^Γ in Y, and assume $H_* \not\leq B$. Then $B \in \mathcal{C}$ by 2.12. Then $B = B \cap Y = B \cap VH = V(B \cap H) = VJ = V$; that is $Y_\Gamma \leq V$.

Let $\gamma \in \Gamma_X$ and suppose $\{\gamma\} \neq \Gamma_X$. Then as J_X^Γ is transitive on Γ_X and Y^Γ is primitive, applying 1.7.3 to Y^Γ and arguing as above in the proof of part (5), using the fact that $Y_\Gamma \le V$, we conclude that Y^Γ is 2-transitive and affine with $F^*(Y^\Gamma) \le V^\Gamma$. Then continuing as in the proof of (5), we obtain a contradiction.

Therefore $\Gamma_X = \{\gamma\}$, so $\Omega_X = \gamma \in \Gamma$. As $H_* \le Y$ and $H_* \not\le V \ge Y_\Gamma$, H_* is not contained in the preimage E of V^Γ in $N_{\hat{G}}(\Gamma)$. Therefore by 2.12, $\bar{E} \in \mathcal{C}$. For $\alpha \in \Gamma$, set $E(\alpha) := E_{\Omega-\alpha}$. Then $X = E(\gamma)$, so as V is transitive on Γ,

$$X^V = \{E(\alpha) : \alpha \in \Gamma\} = \mathcal{X}.$$

Set $F := \langle \mathcal{X} \rangle$. Applying 3.7 to F, FH in the role of Y, X, we conclude that V acts on $\bar{X} \cap F = X(J \cap F)$ and $J_F := J \cap F$. As V acts on XJ_F, $XJ_F = F$. As V and H act on J_F, so does Y. For $\alpha \in \Gamma$, let $\pi_\alpha : F \to E(\alpha)$ be the projection. As $XJ_F = F$, for $\alpha \in \Gamma - \{\gamma\}$, $J_F\pi_\alpha = E(\alpha)$, so as $J_F \trianglelefteq Y$ and V is transitive on Γ, also $X = J_F\pi_\gamma$. Therefore as $J_X \trianglelefteq J_F$, we also have $J_X \trianglelefteq X$. Therefore either $O^2(X) \le J$ or $m_X := |\Omega_X| = 4$ and $J_X = O_2(X)$. By 3.5.2,

$$X = O^2(X)J_X,$$

so as $X \not\le J$, it follows that $O^2(X) \not\le J$. Thus $m_X = 4$ with $O_2(X) = J_X < X$ and $X = O^2(X)J_X$. But now (1) and (2) supply a contradiction.

We have shown that $H_* \le B$. Suppose next that $\Gamma_X \ne \{\gamma\}$. Then

$$Z := X_{\Gamma_X} \in \mathcal{I}_X(H),$$

so by 3.4, $W = \langle Z, V\rangle \in \mathcal{C}$. As WH acts on Γ, it follows from 3.2.2 in [8], that $K_+(\Gamma) \le W$. Let K be the kernel of the action of $N_{\hat{G}}(\Gamma)$ on Γ. As $K/K_+(\Gamma)$ is solvable, $KW \in \mathcal{W}_0$ by 2.13, so $K \le W$. Then as $V \le W$, the preimage B_0 of V^Γ in WH is contained in W. As $H_* \le B \le B_0$, this is a contradiction. Therefore $\Gamma_X = \{\gamma\}$, where $\Omega_X = \gamma \in \Gamma$.

Let $\mathcal{Z} := \{Z \in \mathcal{V}_X(H) : \bar{Z} \in \mathcal{W}_0^*\}$, and for $Z \in \mathcal{Z}$, set $W(Z) := \langle Z, V\rangle$. By 3.4, $W(Z) \in \mathcal{C}$, and by construction, $Z \le W(Z)_\Gamma$, so $W(Z) = W(Z)_\Gamma V$. Then as $V \in \mathcal{W}_0$ and $V_\Gamma = J_\Gamma$, it follows that $\overline{W(Z)_\Gamma} = \bar{Z}$. Thus $W(Z)_\Gamma = ZJ_\Gamma$ is V-invariant. Therefore $XJ_\Gamma = \langle Z : Z \in \mathcal{Z}\rangle J_\Gamma$ is V-invariant, so as $X = E(\gamma)$,

$$F = \prod_{\alpha\in\Gamma} E(\alpha) = \langle X^V \rangle = XJ_\Gamma.$$

Let $P := YF$. Now either $\hat{G} = A$ and $F = K_+(\Gamma)$, or $\hat{G} = S$ and $F = S_\Gamma$. In either case, P_Γ/F is solvable, so $P_\Gamma V \in \mathcal{W}_0$ by 2.13. This is a contradiction as $H_* \le B \le P_\Gamma V$.

This contradiction shows:

10.31. *Assume Hypothesis* 10.24. *Then* $X \le J$.

10.32. *Assume Hypothesis* 10.24 *and let*

$$\mathcal{W}_1 = \{W \in \mathcal{W}_0 : W \not\leq N_G(\Omega_1)\} \quad \textit{and} \quad \mathcal{W}_1^* = \mathcal{W}_1 \cap \mathcal{W}_0^*.$$

Then:

(1) $\mathcal{W}_1^* \neq \varnothing$.

(2) *For each* $W \in \mathcal{W}_1$, *there exists* $U \in \mathcal{W}_1^*$ *with* $U \leq W$.

Proof. Part (1) follows from 3.3. Let $W \in \mathcal{W}_1$. As J acts on Ω, $W \neq J$. Then (2) follows from 3.4. □

10.33. *Assume Hypothesis* 10.24 *and* $X \neq 1$. *Let* $W \in \mathcal{W}_1$, $U \in \mathcal{W}_1^*$ *with* $U \leq W$. *Set* $Y = WH$. *Then:*

(1) X *is transitive on* Ω_X.

(2) W *is transitive on* Ω.

(3) $\Gamma = \Gamma_W := \Omega_X^W$ *is a* X*-invariant partition of* Ω *and* $\Gamma_W = \Gamma_U$.

(4) $K = K_+(\Gamma) \leq J$.

(5) $\hat{G}_\Gamma W \in \mathcal{W}_0$.

(6) Y^Γ *is* 2*-transitive and affine on* Γ, $Y_\Gamma \leq W$, *and* $F^*(Y^\Gamma) \leq W^\Gamma$.

(7) $F^*((WH)^\Gamma) = F^*((UH)^\Gamma)$.

(8) *Let* D_W *be the preimage in* $N_{\hat{G}}(\Gamma)$ *of* $F^*(Y^\Gamma)$. *Then we have* $D_W = D_U$ *and* $\bar{D}_W \in \mathcal{W}_0$.

Proof. As $X \neq 1$ and X contains the group of even permutations in T_2, (1) holds. Then as H is transitive on Ω_1 and W does not act on Ω_1, $Y := WH$ is transitive on Ω. Now by 3.2 in [8], $\mathcal{P}'(Y)$ has a greatest member $\Gamma = \Gamma_W$, $\Omega_X \subseteq \alpha \in \Gamma$, and

$$K = K_+(\Gamma) \leq W.$$

As H is transitive on Ω_1, it follows that $\alpha = \Omega_X$, $1 \neq W^\Gamma$, and Y^Γ is 2-transitive. Therefore W^Γ contains the unique minimal normal subgroup D^Γ of Y^Γ, and D^Γ is transitive. As $X \leq W$ and X is transitive on α, (2) follows. Then

$$\Gamma = \alpha^W = \Omega_X^W.$$

By symmetry between W and U, U is transitive on Ω and $\Gamma_U = \Omega_X^U$. As U is transitive on Ω, it is also transitive on Γ, so $\Gamma = \Omega_X^U$, and hence $\Gamma = \Gamma_U$, completing the proof of (3).

Again by symmetry between U and W, $K \leq U$. As J is not transitive on Γ but U is transitive, $J^\Gamma \neq U^\Gamma$, so $U_\Gamma \leq J$. Thus $K \leq U_\Gamma \leq J$, establishing (4).

As $\hat{G}_\Gamma/K$ and Y_Γ/K are 2-groups, $W\hat{G}_\Gamma$ and WY_Γ are in $\mathcal{W}_0$ by 2.13, so (5) holds. Then $WY_\Gamma = WH \cap WY_\Gamma = W(H \cap WY_\Gamma) = WJ = W$, so $Y_\Gamma \le W$.

If Y^Γ is almost simple, then

$$H_*^\Gamma \le D^\Gamma \le W^\Gamma,$$

so as $Y_\Gamma \le W$, $H_* \le W$, a contradiction. Thus (6) holds as Y^Γ is 2-transitive.

Part (7) follows from (3), (6), and symmetry between U and W. Then (8) follows from (5) and (7). □

10.34. *Assume Hypothesis* 10.24. *Then* $X = 1$.

Proof. Assume otherwise. By 10.32.1, there is $U \in \mathcal{W}_1^*$. Adopt the notation of 10.33, and let $\Gamma := \Gamma_U$ and $D := D_U$. Let $\mathcal{C}$ be the connected component of Λ' containing U, and $\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*$. Then for each $U \le W \in \mathcal{W}_0$, $W \in \mathcal{C} \cap \mathcal{W}_1$, and by 10.33, $\Gamma = \Gamma_W$ and $D = D_W$. But for each $V \in \mathcal{C}^*$, $U \le \langle U, V\rangle \in \mathcal{W}_0$, so $V \le W \le N_G(D)$. Thus $\mathcal{C} \subseteq N_G(D)$, so 3.8 supplies a contradiction. □

10.35. *Assume Hypothesis* 10.24. *Then one of the following holds:*

(1) $k \le 3$, $|\Omega_X| = 2$, *and* $\hat{G} = A$.

(2) $k = 2$ *and* $|\Omega_X| = 1$.

Proof. By 10.34, $X = 1$, so $|\Omega_X| \le 2$, and in case of equality, $\hat{G} = A$. As $k \le 3$ by 10.5, the lemma follows. □

10.36. *Assume Hypothesis* 10.24. *Let* $W \in \mathcal{W}_1$, *and let* Ω_W *be the orbit of*

$$Y := WH$$

on Ω *containing* Ω_1. *Then one of the following holds:*

(1) $|\Omega_W| = |\Omega_1| + 1$, *and* Y *is* 2-*transitive and affine on* Ω_W. *Thus*

$$F^*(Y) =: D_W$$

is regular on Ω_W, *and* H *is transitive on* $D_W^\#$ *via conjugation. Further*

$$\bar{D}_W \in \mathcal{W}_0^*.$$

(2) $|\Omega_X| = 2$, $\Omega_W = \Omega$, $\Gamma = \Gamma_W = \Omega_X^W$ *is a* Y-*invariant partition of* Ω, $\hat{G} = A$, $\hat{G}_\Gamma J \in \mathcal{W}_0 - \mathcal{W}_1$, $\hat{G}_\Gamma W \in \mathcal{W}_0$, $Y_\Gamma \le W$, *and* Y^Γ *is* 2-*transitive and affine. Let* D_W *be the preimage in* W *of* $F^*(Y^\Gamma)$. *Then* $\bar{D}_W \in \mathcal{W}_0'$ *and* H^Γ *is transitive on* $D^{\Gamma\#}$.

(3) *$k = 2$, $\Omega_X = \{\alpha_1, \alpha_2\}$ is of order 2, $\Omega_W = \Omega$, H_{α_1} has two orbits θ_i, $i = 1, 2$, on Ω_1, and setting*

$$\Xi_i := \theta_i \cup \{\alpha_i\} \quad and \quad M := N_{WH}(\Xi_1),$$

$\Xi_W := \{\Xi_1, \Xi_2\} \in \mathcal{P}'(Y)$, $\hat{G} = A$, and M^{Ξ_i} is 2-transitive and affine. Let D_i be the projection of $F^*(M^{\Xi_i})$ on $G_{\Omega - \Xi_i}$ and $D_W := D_1 D_2$. Then we have $\bar{D}_W \in \mathcal{W}_0'$ and $F^*(Y)$ is D_W or a full diagonal subgroup of D_W.

Proof. As $W \in \mathcal{W}_1$, Ω_1 is a proper subset of $\Sigma := \Omega_W$. But by 10.21, $|\Omega_X| \leq 2$, so either $|\Sigma| = |\Omega_1| + 1$, or $|\Omega_X| = 2$ and $\Sigma = \Omega$. Assume the first case holds. Then as H is transitive on Ω_1, Y is 2-transitive on Σ, so by 2.15, Y is affine on Σ, and setting $D_W := F^*(Y)$, $D_W \leq W$ with $\bar{D}_W \in \mathcal{W}_0$. As H is transitive on Ω_1, H is transitive on $D_W^\#$ by conjugation. In particular $\bar{D}_W \in \mathcal{W}_1$ and H is maximal in $\bar{D}H = DH$, so we have $\bar{D}_W \in \mathcal{W}_0^*$. Thus (1) holds in this case.

So assume the second case holds. Suppose $\Gamma = \Omega_X^W$ is a partition of Ω, and let $K = \hat{G}_\Gamma$. As K is an H-invariant 2-group, $\bar{K}$ and KW are in $\mathcal{W}_0$ by 2.13, and by construction $\bar{K} = KJ$ acts on Ω_1, so $\bar{K} \notin \mathcal{W}_1$. As $X = 1$ by 10.34, $\hat{G} = A$. As $Y_\Gamma \leq K$ and $KW \in \mathcal{W}_0$, $WY_\Gamma \in \mathcal{W}_0$ by 2.7, so

$$WY_\Gamma = HW \cap WY_\Gamma = W(H \cap WY_\Gamma) = WJ = W.$$

That is $Y_\Gamma \leq W$.

As $\bar{K} \notin \mathcal{W}_1$, $W^\Gamma \neq 1$. As Y is transitive on Ω and H is transitive on $\Gamma - \{\Omega_X\}$, Y^Γ is 2-transitive. Then as $W^\Gamma \neq 1$ and $Y_\Gamma \leq W$, Y^Γ is affine by 2.15. Define $D = D_W$ as in (2). Then $\bar{D} \in \mathcal{W}_0$ by 2.15. As D does not act on Ω_1, $\bar{D} \in \mathcal{W}_0'$. As H is transitive on Ω_1, H^Γ is transitive on $D^{\Gamma\#}$. Thus (2) holds in this case.

Thus we may assume Γ is not a partition of Ω. It follows from 1.6 that case (i) or (ii) of 1.6.1 holds. Suppose that case (i) holds. Then Y is $5/2$-transitive on Ω. By 2.14, Y is affine, so by 1.6.2, $D := F^*(Y) \cong E_{2^e}$ is regular on Ω, and for $\omega \in \Omega_X$, Y_ω is isomorphic to $L_e(2)$, or possibly A_7 if $e = 4$. Then as Y/W is almost simple, we conclude $W = D$ and $Y_\omega = H = H_*$. But now H is transitive on $\Omega - \{\omega\}$, whereas H acts on Ω_X of order 2, a contradiction.

Thus we may assume case (ii) holds. Then, together with 1.6.3, this implies that (3) holds, except for the condition that $\bar{D}_W \in \mathcal{W}_0$, which follows from 2.13. □

10.37. *$k = 2$ and $I_*' = \varnothing$.*

Proof. Assume otherwise. If $k = 2$, we may assume I_*' is nonempty, so Hypothesis 10.24 is satisfied. On the other hand if $k \neq 2$, then by 10.29, $k = 3$. Then from the discussion in 10.25, Hypothesis 10.24 is satisfied.

So in any event, Hypothesis 10.24 is satisfied. By 10.32.1 we may choose

$$W \in \mathcal{W}_1.$$

Define Ω_W and $Y := WH$ as in 10.36. By 10.36 one of the three conclusions of that lemma are satisfied. Let $\mathcal{C}$ be the connected component of Λ' containing W and $\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*$.

Suppose first that conclusion (1) of 10.36 is satisfied, and set $D := D_W$. Then $\bar{D} \in \mathcal{W}_0^*$ by 10.36. Let $D \leq Z \in \mathcal{W}_0$. As D does not act on Ω_1, $Z \in \mathcal{W}_1$. We claim $\Omega_W = \Omega_Z$, so Z satisfies conclusion (1) of 10.36. If not $\Omega_Z =: \Omega$ is of order $|\Omega_1| + 2$, so as DH is 2-transitive on Ω_W of order $|\Omega_1| + 1$, ZH is 3-transitive on Ω, impossible as ZH is imprimitive on Ω in cases (2) and (3) of 10.36. Thus the claim is established.

By the claim, $\Omega_Z = \Omega_{\bar{D}} = \Omega_W$, and then as $DH \leq ZH$,

$$D = F^*(DH) = F^*(ZH) =: D_Z.$$

Let $V \in \mathcal{C}^*$ and $Z(V) = \langle D, V \rangle$. As $\bar{D} \in \mathcal{W}_0^*$, it follows that $Z(V) \in \mathcal{C}$ by 3.4, so $D = D_{Z(V)}$. Thus $V \leq N_G(D)$ for each $V \in \mathcal{C}^*$, so $\mathcal{C} \leq N_G(D)$, contrary to 3.8. Thus no member of $\mathcal{W}_1$ satisfies conclusion (1) of 10.36.

Next suppose W satisfies conclusion (2) of 10.36. Adopt the notation in (2) and let $K := \hat{G}_\Gamma$ and $D := D_W$. Let $\tau \in S$ be the involution with cycles (ω, ω'), for $\{\omega, \omega'\} \in \Gamma$, and set $T = \langle \tau \rangle$. Let $\tau' = \tau \cdot (\omega_1, \omega_2)$, where $\Omega_X = \{\omega_1, \omega_2\}$.

Suppose first that $B \in \mathcal{J}_K(H)$ with $B \not\leq T$. Claim there in no $\Xi \in \mathcal{P}(BH)$ with $\Xi = \{\Xi_1, \Xi_2\}$ of order 2. For if $\gamma \in \Gamma' = \Gamma - \{\Omega_X\}$ with $\gamma \subseteq \Xi_1$, then as H is transitive on Γ', it follows that $\Omega_1 \subseteq \Xi_1$, contradicting

$$|\Xi_1| = n/2 < n - 2 = |\Omega_1|.$$

We conclude for each $\gamma \in \Gamma'$, $|\Xi_i \cap \gamma| = 1$ for $i = 1, 2$, and then also

$$|\Xi_i \cap \Omega_X| = 1.$$

Let $b \in B - T$. Then there are $\alpha, \beta \in \Gamma$ with $\alpha \subseteq M(b)$ and $\beta \subseteq \mathrm{Fix}(b)$. As $\alpha \subseteq M(b)$ and $|\Xi_1 \cap \alpha| = 1$, $\Xi_1 b = \Xi_2$. This is impossible as b fixes the member of $\Xi_1 \cap \beta$, establishing the claim.

Suppose next that for each $Z \in \mathcal{W}_0$ containing W, ZH satisfies conclusion (2) of 10.36; for example if $W_\Gamma \not\leq T$ this holds by the previous paragraph. Let

$$W \leq Z \in \mathcal{W}_0.$$

As W does not act on Ω_1, it follows that $Z \in \mathcal{W}_1$. By hypothesis, Z satisfies conclusion (2) of 10.36. Then $\Gamma_Z = \Omega_X^Z = \Omega_X^W = \Gamma$, so $K = \hat{G}_{\Gamma_Z}$ and then by Proposition 4 in [4], $D = D_Z$. Thus $Z \leq N_G(D)$, and we obtain a contradiction as in the previous paragraph unless $W \in \mathcal{W}_0^!$.

By 10.36, we have $\bar{D} \in \mathcal{W}_0'$. By 3.7 there is $D_0 \leq \mathcal{V}_D(H)$ such that $\bar{D}_0 \in \mathcal{W}_0^*$ and $D_0^\Gamma \neq 1$. As H is irreducible on D^Γ, it follows that $D_0^\Gamma = D^\Gamma$, so $\bar{D}_0$ satisfies conclusion (2) of 10.36 with $D_{D_0} = D$. Replacing W by $\bar{D}_0$, we may assume $W = \bar{D}_0 \in \mathcal{W}_0^*$. Then $W \notin \mathcal{W}_0^!$, so $W_\Gamma \le T$ by the previous paragraph. In particular $J_\Gamma \le T$.

By 10.36, $WK \in \mathcal{W}_0$, so by 3.7 there is $K_0 \in \mathcal{V}_K(H)$ with $K_0 \not\le T$, $\bar{K}_0 \in \mathcal{W}_0^*$, and $Z := \langle W, K_0 \rangle \in \mathcal{W}_0$. By the claim, Z acts on no partition of order 2, so we get $Z \in \mathcal{W}_0^!$. Thus as $W, \bar{K}_0 \in \mathcal{C}^*$, $\mathcal{C}^* = \{W, \bar{K}_0, V\}$ is of order 3. As $KW \in \mathcal{C}$, it follows that $\bar{K} = \bar{K}_0$, so $K = K_0 J_\Gamma \le K_0 T$. In particular H is irreducible on KT/T, so it follows that $n/2$ is even (or else $\tau' \in C_K(H)$) and hence $\tau \in A$, so that $T \le K$ is H-invariant. Then $T = J_\Gamma$ by 3.5.3.

By 3.8, V does not act on Γ. Next $U := \langle V, W \rangle \in \mathcal{C}$, and as V does not act on Γ, by an earlier reduction, UH does not satisfy conclusion (2) of 10.36, so UH satisfies conclusion (3) of 10.36. Thus $M := N_{UH}(\Xi_1)$ is of index 2 in UH, and $D = D_M \times T$, where $D_M := D \cap M \cong D^\Gamma$. Then $D_M \cong D_M^{\Xi_1} = F^*(M^{\Xi_1})$. As in 10.36.3, let D_i be the projection of D_M on $G_{\Omega - \Xi_i}$ and $D_U := D_1 D_2$. Now $\bar{D}_U \in \mathcal{W}_0$ by 10.36 and $Y = C_{UHD_U}(\tau)$, so $W = \bar{D}_M < \bar{D}_W$ and $J \cap D_U = 1$. Thus $U = \bar{D}_U$ so as $V \in \mathcal{W}_0^*$, $V = \bar{D}_V$, where $D_V := V \cap D_U$. Thus by 3.4 and 3.7, $UH = D_U H$ and $D_U = D_M \times D_V$. Finally $\tau \in J$, so τ acts on D_V. However as τ centralizes H and acts on V, and as H is irreducible on D_V,

$$D_V \le C_{D_U}(\tau) = D_M,$$

a contradiction.

Therefore conclusion (3) of 10.36 is satisfied for each $W \in \mathcal{W}_1$. Set

$$Y_W := WH.$$

Then we have $\Xi_W = \{\Xi_1, \Xi_2\}$ with $\Xi_i = \Xi_{W,i}$ the orbits of $F^*(M_W)$ on Ω, where $M_W := N_{Y_W}(\Xi_1)$, and $M_W^{\Xi_i}$ is affine.

Pick $W \in \mathcal{W}_0^*$ and let $W \le Z \in \mathcal{W}_0$. As $W \in \mathcal{W}_0^*$, it follows that $W = \bar{E}$, where $E := W \cap D_W$, and $D_W = D_1 \times D_2$ is defined in 10.36.3. As E is contained in each subgroup of Y_W of index 2, $E \le M_Z$. Thus E acts on $\Xi_{Z,i}$ of order $n/2$, so as the orbits $\Xi_{W,i}$ of E are of order $n/2$, it follows that $\Xi_Z = \Xi_W = \Xi$. By Proposition 4 in [4], $E^{\Xi_i} = F^*(M_Z^{\Xi_i})$, so $D = D_W = D_Z$.

Let $\mathcal{C}$ be the connected component of Λ' containing W, and $\mathcal{C}^* = \mathcal{C} \cap \mathcal{W}_0^*$. For each $V \in \mathcal{C}^*$, $W \le Z = \langle V, W \rangle \in \mathcal{C}$, so $V \le Z \le N_G(D)$ by the previous paragraph. This contradicts 3.8, and completes the proof. □

10.38. *Set $H_i := H_* \pi_i$ and $J_i := J_* \pi_i$. Then:*

(1) *$k = 2$, $I_* = \{1, 2\}$, $m_i = |\Omega_i| \ge 5$, and $H_i / J_i \cong L$.*

(2) *$\bar{H}_i$ and $\bar{X}_i$ are in $\mathcal{W}_0'$.*

(3) *$H_i = F^*(X_i)$.*

(4) *$X_i \in \mathcal{W}_0^*$.*

Proof. By 10.37, $k = 2$ and $I_* = \{1, 2\}$, so (1) follows from part (2) of 10.3.

As $I_* = \{1, 2\}$, $i \in \mathcal{S}$, so $\bar{X}_i \in \mathcal{W}_0$ by 10.3.3. As $J_i \neq H_i$, we have $H_i \not\leq J$, so $\bar{H}_i \neq J$, and hence (2) holds.

Let $Y_i = F^*(X_i)$. As $m_i \geq 5$, $Y_i \cong A_{m_i}$ is nonabelian simple and of index at most 2 in X_i. By (2) and 2.7, $\bar{Y}_i \in \mathcal{W}_0$, and then by 3.5.2, $\bar{X}_i = \bar{Y}_i$. If $\bar{X}_i \in \mathcal{W}_0^*$, then (4) holds and (3) follows from (2), so we may assume otherwise. Then by 3.7 there exists $Z \in \mathcal{V}_{X_i}(H)$ with $\bar{Z} \in \mathcal{W}_0'$, $X_i = \langle H_i, Z \rangle$, and $H_i \cap Z \leq J$. As Z is H_*-invariant, H_i acts on Z, so $Z \trianglelefteq \langle H_i, Z \rangle = X_i$, and hence $Y_i \leq Z$, contradicting $Z \cap H_i \leq J$. This completes the proof of (3) and (4). □

Theorem 10.39. *Assume Hypothesis* 5.1. *Then H is transitive on* Ω.

Proof. Assume otherwise. Then Hypothesis 10.1 is satisfied, so we can appeal to the results in this section. Adopt the notation of 10.38, and choose notation so that $m_1 \geq m_2$. Let $\mathcal{C}$ be the connected component of Λ' containing $\bar{X}_1$ and

$$\mathcal{C}^* := \mathcal{C} \cap \mathcal{W}_0^*.$$

By 3.8, there exists $V \in \mathcal{C}^*$ such that $V \not\leq N_G(\Omega_1)$. By 10.38.4, $\bar{X}_1 \in \mathcal{W}_0^*$, so by 3.4,

$$W := \langle X_1, V \rangle \in W_0.$$

Set $Y := WH$. As H is transitive on Ω_i and $Y \not\leq N_G(\Omega_1)$, it follows that Y is transitive on Ω. Then as $X_1 \leq W$ and $m_1 \geq 5$, it follows from 3.2 in [8] that there is $\Gamma \in \mathcal{P}'(Y)$ with $\Omega_1 \subseteq \gamma \in \Gamma$ and $K_+(\Gamma) \leq W$. As $m_1 \geq m_2$, it follows that $m_1 = n/2$ and $\Gamma = \{\Omega_1, \Omega_2\}$. But now

$$O^2(X_2) \leq K_+(\Gamma) \leq W,$$

so $H_* \leq X_1 O^2(X_2) \leq W$, contradicting $W \in \mathcal{W}_0$. This contradiction completes the proof of the theorem. □

Bibliography

[1] M. Aschbacher, *Finite Group Theory*, Cambridge University Press, Cambridge, 1986.

[2] M. Aschbacher, Projective planes, loops, and groups, *J. Algebra* **300** (2006), 396–432.

[3] M. Aschbacher, On intervals in subgroup lattices of finite groups, *J. Amer. Math. Soc.* **21** (2008), 809–830.

[4] M. Aschbacher, Overgroups of primitive groups, *J. Aust. Math. Soc.* **87** (2009), 37–82.

[5] M. Aschbacher, Overgroups of primitive groups, II, *J. Algebra* **322** (2009), 1586–1626.

[6] M. Aschbacher, Signalizer lattices in finite groups, *Michigan J. Math.* **58** (2009), 79–103.

[7] M. Aschbacher and L. Scott, Maximal subgroups of finite groups, *J. Algebra* **92** (1985), 44–80.

[8] M. Aschbacher and J. Shareshian, Restrictions on the structure of subgroup lattices of alternating and symmetric groups, *J. Algebra* **322** (2009), 2449–2463.

[9] M. Aschbacher and S. Smith, *The Classification of Quasithin Groups*, Mathematical Surveys and Monographs 11, American Mathematical Society, Providence, RI, 2004.

[10] R. Baddeley and A. Lucchini, On Representing finite lattices as intervals in subgroup lattices of finite groups, *J. Algebra* **196** (1997), 1–100.

[11] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups. Number 3*, Mathematical Surveys and Monographs 40, American Mathematical Society, Providence, RI, 1999.

[12] M. Liebeck, C. Praeger and J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, *J. Algebra* **111** (1987), 365–383.

[13] P. Palfy and P. Pudlak, Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups, *Algebra Universalis* **11** (1980), 22–27.

[14] P. Perepelitsky, Intervals in subgroup lattices of the alternating and symmetric groups of prime degree, *J. Group Theory* **12** (2009), 119–137.

[15] J. Shareshian, Topology of order complexes of intervals in subgroup lattices, *J. Algebra* **268** (2003), 677–686.

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